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Pseudo-Jacobian and global inversion of nonsmooth mappings on Riemannian manifolds

E. Ghahraei[a], S. Hosseini[b], M.R. Pouryayevali[a,*]

[a] *Department of Mathematics, University of Isfahan, P. O. Box 81745-163, Isfahan, Iran*
[b] *ETH Zürich, Seminar for Applied Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland*

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We introduce a notion of pseudo-Jacobian for continuous nonsmooth mappings between Riemannian manifolds. Then we discuss the global inversion of continuous mappings between Riemannian manifolds which may be non-locally Lipschitz. For this purpose, we obtain a version of the Hadamard integral condition for invertibility by using pseudo-Jacobian.

1. Introduction

In 1906, the following criterion for the global invertibility of C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ was established by Hadamard. Assume that the differential $df(x)$ is invertible for every x in $\mathbb{R}^n$ and let

$$m(t) := \inf\{(1/\|df(x)^{-1}\|) : \|x\| \leq t\}.$$

If $\int_0^\infty m(t)dt = \infty$, then f maps $\mathbb{R}^n$ diffeomorphically onto $\mathbb{R}^n$.

This result has been extended in various contexts; for instance we refer to [9,14] for extensions of Hadamard integral condition to the case of mappings between Banach spaces. The global invertibility of local diffeomorphisms between Banach–Finsler manifolds has been investigated in [16]. The global inversion results in metric spaces by using a version of Hadamard integral condition has been studied in [2,3].

In the case when f is only locally Lipschitz continuous, then the differential $df(x)$ may not exist, but one can replace it by the Clarke generalized Jacobian $\partial f(x)$, which in the finite-dimensional case, is a nonempty, compact, convex subset of linear transformations of $\mathbb{R}^n$. In [15] the author extended the Hadamard's theorem to the class of locally Lipschitz continuous mappings on $\mathbb{R}^n$.

* Corresponding author.
E-mail addresses: e.ghahraei@sci.ui.ac.ir (E. Ghahraei), seyedehsomayeh.hosseini@math.ethz.ch (S. Hosseini), pourya@math.ui.ac.ir (M.R. Pouryayevali).

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This result has been extended in [6] to the global inversion of continuous nonsmooth mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ which may be non-locally Lipschitz, for which the Clarke generalized Jacobian is not defined. The authors used the concept of pseudo-Jacobian map associated to f introduced in [7] and obtained a version of the Hadamard integral condition. Then a sufficient condition for global inversion was deduced.

A manifold in general does not have a linear structure, therefore to study the problem of global inversion of nonsmooth mappings on manifolds new techniques are needed. In the past few years, a number of results have been obtained on various aspects of nonsmooth analysis on Riemannian manifolds; see, e.g. [1,11,4]. In [5] the authors extend the definition of the Clarke generalized Jacobian to locally Lipschitz mappings on manifolds and generalize the results in [15] to Finsler manifolds.

Our main purpose in this paper is to present a notion of pseudo-Jacobian for continuous mapping, which are not necessarily locally Lipschitz, between Riemannian manifolds. We use this notion to obtain sufficient conditions for a continuous mapping between Riemannian manifolds to be a global homeomorphism.

The paper is organized as follows. In Section 2 we introduce the notion of pseudo-Jacobian associated to a continuous mapping between Riemannian manifolds and some basic properties are presented. In Section 3, for a mapping $f : M \to N$ we introduce the regularity index of f related to a pseudo-Jacobian mapping $\partial^* f$ and we discuss its relation with the lower scalar Dini derivative. In Section 4 we prove our main results. Considering continuous mappings between Riemannian manifolds, we obtain a version of Hadamard integral condition using the regularity index which provides a sufficient condition for global inversion.

2. Pseudo-Jacobian of continuous functions

Let us first introduce some standard notations and known results of Riemannian manifolds; see, e.g. [10,18]. Throughout this paper all manifolds are complete connected finite-dimensional and endowed with the Riemannian metric $\langle ., . \rangle_x$ on the tangent space at each point x. The corresponding norm is denoted by $\| \|_x$. As usual we denote by $B(x, \delta)$ the open ball with respect to the Riemannian distance on M centered at x with radius δ. In linear spaces $\operatorname{co} A$ denotes the convex hull of a subset A.

Recall that the set S in a Riemannian manifold M is called convex if every two points $p_1,\ p_2 \in S$ can be joined by a unique geodesic whose image belongs to S. We identify (via the Riemannian metric) the tangent space of M at a point x, denoted by T_xM, with the cotangent space at x, denoted by T_x^*M. For the point $x \in M$, $\exp_x : U_x \to M$ will stand for the exponential function at x, where U_x is an open subset of T_xM. Recall that $\exp_x$ maps straight lines of the tangent space T_xM passing through $0_x \in T_xM$ into geodesics of M passing through x.

To define the notion of pseudo-Jacobian associated to a continuous mapping $f : M \to N$, we have to work with the vector bundle $\mathcal{L}(TM, f^*TN)$. Let M and N be smooth manifolds of dimension m and n, respectively, and let $f : M \to N$ be a continuous mapping. We consider the vector bundle $\mathcal{L}(TM, f^*TN)$ defined as the following disjoint union:

$$\begin{aligned}\mathcal{L}(TM, f^*TN) &:= \bigsqcup_{x\in M} \mathcal{L}(T_xM, T_{f(x)}N)\\ &= \{(x, B) : x \in M \quad \text{and} \quad B \in \mathcal{L}(T_xM, T_{f(x)}N)\},\end{aligned}$$

where $\mathcal{L}(T_xM, T_{f(x)}N)$ denotes the space of all linear mappings from T_xM to $T_{f(x)}N$ and the bundle projection $\pi : \mathcal{L}(TM, f^*TN) \to M$ is given by $\pi(x, B) = x$; see for instance [5,13] for details.

Note that the topology of $\mathcal{L}(TM, f^*TN)$ is generated by the basic open sets $\mathcal{O}(U, V, W)$ defined as

$$\mathcal{O}(U, V, W) := \bigsqcup_{x\in U} W_{x,\varphi,\psi} = \{(x, B) : x \in U \text{ and } B \in W_{x,\varphi,\psi}\},$$