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Antimaximum principle in exterior domains

T.V. Anoop[a], P. Drábek[b,*], Lakshmi Sankar[c], Sarath Sasi[c]

[a] Department of Mathematics, Indian Institute of Technology Madras, Chennai 36, India

[b] Department of Mathematics and NTIS, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 22, 306 14 Plzeň, Czech Republic

[c] School of Mathematical Sciences, National Institute of Science Education and Research, Jatni, Odisha, PIN 752050, India

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ABSTRACT

We consider the antimaximum principle for the p-Laplacian in the exterior domain:

$$\begin{cases} -\Delta_p u = \lambda K(x) \mid u \mid^{p-2} u + h(x) & \text{in } B_1^c, \\ \quad u = 0 \quad \text{on } \partial B_1, \end{cases}$$

where Δ_p is the p-Laplace operator with $p > 1, \lambda$ is the spectral parameter and B_1^c is the exterior of the closed unit ball in $\mathbb{R}^N$ with $N \geq 1$. The function h is assumed to be nonnegative and nonzero, however the weight function K is allowed to change its sign. For K in a certain weighted Lebesgue space, we prove that the antimaximum principle holds locally. A global antimaximum principle is obtained for h with compact support. For a compactly supported K, with $N = 1$ and $p = 2$, we provide a necessary and sufficient condition on h for the global antimaximum principle. In the course of proving our results we also establish the boundary regularity of solutions of certain boundary value problems.

1. Introduction

In this paper, we are concerned with the antimaximum principle for the following quasilinear problem:

$$\begin{aligned} &-\Delta_p u = \lambda K(x)|u|^{p-2}u + h(x) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Δ_p is the p-Laplace operator with $p > 1,\ \lambda$ is the spectral parameter and Ω is a domain in $\mathbb{R}^N$ with $N \geq 1$. In particular, we will be considering $\Omega = B_1^c$, the exterior of the closed unit ball centered at the

* Corresponding author. Tel.: +420 377632648.
E-mail addresses: anoop@iitm.ac.in (T.V. Anoop), pdrabek@kma.zcu.cz (P. Drábek), lakshmi@niser.ac.in (L. Sankar), sarath@niser.ac.in (S. Sasi).

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origin. The weight function K is allowed to change sign and $h \in L^\infty_{\text{loc}}(\Omega)$ is a nonnegative nonzero source term that lies in a suitable function space. Further assumptions on the functions K and h will be specified later. Through out this article, the solutions are understood in a weak sense (see Definition 1.2).

Let Ω be a bounded domain in $\mathbb{R}^N$. Let λ_1 and λ_2 be the first and second eigenvalues of the Laplacian on Ω with the Dirichlet boundary condition. For $h \in L^2(\Omega)$ and for $\lambda \in (-\infty, \lambda_2) \setminus \{\lambda_1\}$, the Fredholm alternative for self adjoint compact operators ensures the existence of solutions for the following problem:

$$\begin{aligned} &-\Delta u = \lambda u + h(x) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

For $\lambda < \lambda_1$ and $h \geq 0 (\not\equiv 0)$, it can be easily verified that the solutions of (2) are nonnegative. Furthermore, the strong maximum principle implies that any nonnegative solution of (2) is strictly positive in Ω. On the other hand, for certain $\lambda > \lambda_1$, Clement and Peletier (in [4]) observed that a complete opposite of the above phenomena happens, i.e., there exists $\delta(h) > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ any solution of (2) is completely negative in Ω. They named this phenomena as the antimaximum principle (in short AMP). Thereafter many versions and generalizations of the AMP were proved for both Laplacian and p-Laplacian on bounded domains. For example, see [3,8,9,13] and the references therein.

On an unbounded domain Ω and for p-Laplacian with $p \neq 2$, the right analogue of (2) is given by (1). Furthermore, in this case, the eigenvalue λ_1 is given by the following weighted eigenvalue problem associated to (1):

$$\begin{cases} -\Delta_p u = \lambda K(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3}$$

The AMP for (1) was studied in [6,14] for the weight function K that belongs to certain classes of Lebesgue spaces. Unlike in the case of bounded domains, the AMP does not hold for (1) on unbounded domains in general. However, in [6,14] it is shown that a local version of AMP holds for (1).

Definition 1.1. For a given $h \geq 0, h \not\equiv 0$, we say that a *local* AMP holds for (1), if for any bounded set $E \subset \Omega$ there exists $\delta = \delta(h, E)$ such that any solution u_λ of (1) is negative in E for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$. If there exists a $\delta = \delta(h)$ such that u_λ is negative in Ω for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$ then we say that *global* AMP holds for (1).

In [14], the authors consider $\Omega = \mathbb{R}^N$ with $1 < p < N$ and K satisfying the following conditions

(i) $K \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$ and $\text{supp}(K^+)$ is of positive measure,
(ii) $K = g_1 + g_2 - g_3 - g_4$ such that $g_i \geq 0$, $g_1 \in L^{\frac{N}{p}}(\mathbb{R}^N)$ and $g_i \in L^\infty(\mathbb{R}^N)$ for $i = 1, 2, 3, 4$.

Under some additional conditions on g_2, g_3 and g_4 (see H_1 and H_2 of [14]), the existence of the first eigenvalue λ_1 for (3) is proved in [14]. Furthermore, for a given $h \in L^\infty(\mathbb{R}^N)$ such that $h \geq 0, h \not\equiv 0$ they prove a local AMP.

The local AMP for (1) with $\Omega = \mathbb{R}^N$ has been extended for $p \geq N$ in [6] with the restriction that the weight function K has dominant negative part at infinity. More precisely, the authors considered K of the form $K = g_1 - g_2$ with $g_1, g_2 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and satisfies the following conditions:

(i) for $p < N$, $g_1 \in L^{\frac{N}{p}}(\mathbb{R}^N)$ and $g_1,\ g_2 \geq 0$,
(ii) for $p \geq N$, there exist an integer $N_0 > p$ and $\varepsilon_0 > 0$ such that $g_1 \in L^{\frac{N_0}{p}}(\mathbb{R}^N)$, $g_1 \geq 0$ and $g_2 \geq \varepsilon_0$,
(iii) $\text{supp}(K^+)$ has a positive measure.