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Besov regularity of solutions to the p-Poisson equation

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ABSTRACT

In this paper, we are concerned with regularity analysis for solutions to nonlinear partial differential equations. Many important practical problems are related with the p-Laplacian. Therefore, we are particularly interested in the smoothness of solutions to the p-Poisson equation. For the full range of parameters $1 < p < \infty$ we investigate regularity estimates in the adaptivity scale $B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$, of Besov spaces. The maximal smoothness σ in this scale determines the order of approximation that can be achieved by adaptive and other nonlinear approximation methods. It turns out that, especially for solutions to p-Poisson equations with homogeneous Dirichlet boundary conditions on bounded polygonal domains, the Besov regularity is significantly higher than the Sobolev regularity which justifies the use of adaptive algorithms. This type of results is obtained by combining local Hölder with global Sobolev estimates. In particular, we prove that intersections of locally weighted Hölder spaces and Sobolev spaces can be continuously embedded into the specific scale of Besov spaces we are interested in. The proof of this embedding result is based on wavelet characterizations of Besov spaces.

1. Introduction

This paper is concerned with regularity estimates of the solutions to the *p-Poisson equation*

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f \quad \text{in } \Omega, \tag{1}$$

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where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^d$ denotes some bounded Lipschitz domain. The corresponding variational formulation is given by

$$\int_\Omega \left\langle |\nabla u|^{p-2} \nabla u, \nabla v \right\rangle \mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega). \tag{2}$$

Problems of this type arise in many applications, e.g., in non-Newtonian fluid theory, non-Newtonian filtering, turbulent flows of a gas in porous media, rheology, radiation of heat and many others. Moreover, the p-Laplacian has a similar model character for nonlinear problems as the ordinary Laplace equation for linear problems. We refer to [37] for an introduction. By now, many results concerning existence and uniqueness of solution are known, we refer again to [37] and the references therein. However, in many cases, the concrete shape of the solutions is unknown, so that efficient numerical schemes for the constructive approximation are needed. In practice, e.g., for problems in three and more space dimensions, this might lead to systems with hundreds of thousands or even millions of unknown. Therefore, a quite natural idea would be to use *adaptive* strategies to increase efficiency. Essentially, an adaptive algorithm is an updating strategy where additional degrees of freedom are only spent in regions where the numerical approximation is still “far away” from the exact solution. Nevertheless, although the idea of adaptivity is quite convincing, these schemes are hard to analyze and to implement, so that some theoretical foundations that justify the use of adaptive strategies are highly desirable.

The analysis in this paper is motivated by this problem, in particular in connection with adaptive wavelet algorithms. In the wavelet case, there is a natural benchmark scheme for adaptivity, and that is best n-term wavelet approximation. In best n-term approximation, one does not approximate by linear spaces but by nonlinear manifolds $\mathcal{M}_n$, consisting of functions of the form

$$S = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda, \tag{3}$$

where $\{\psi_\lambda \mid \lambda \in \mathcal{J}\}$ denotes a given wavelet basis and $\Lambda \subset \mathcal{J}$ with $\#\Lambda = n$. We refer to Section 2 and to the textbooks [13,40,51] for further information concerning the construction and the basic properties of wavelets. In the wavelet setting, a best n-term approximation can be realized by extracting the n biggest wavelet coefficients from the wavelet expansion of the (unknown) function one wants to approximate. Clearly, on the one hand, such a scheme can never be realized numerically, because this would require to compute all wavelet coefficients and to select the n biggest. On the other hand, the best we can expect for an adaptive wavelet algorithm would be that it (asymptotically) realizes the approximation order of the best n-term approximation. In this sense, the use of adaptive schemes is justified if best n-term wavelet approximation realizes a significantly higher convergence order when compared to more conventional, uniform approximation schemes. In the wavelet setting, it is known that the convergence order of uniform schemes with respect to L_p depends on the regularity of the object one wants to approximate in the scale $W^s(L_p(\Omega))$ of L_p-Sobolev spaces, whereas the order of best n-term wavelet approximation in L_p depends on the regularity in the *adaptivity scale* $B^\sigma_\tau(L_\tau(\Omega)), 1/\tau = \sigma/d + 1/p$, of Besov spaces. We refer to [7,14, 26] for further information. Therefore, the use of adaptive (wavelet) algorithms for (1) would be justified if the Besov smoothness σ of the solution in the adaptivity scale of Besov spaces is higher than its Sobolev regularity s.

For linear second order elliptic equations, a lot of positive results in this direction already exist; see, e.g., [6,8,10]. In contrast, it seems that not too much is known for nonlinear equations. The only contribution we are aware of is the paper [11] which is concerned with semilinear equations. In the present paper, we show a first positive result for quasilinear elliptic equations, i.e., for the p-Poisson equation (1). Results of Savaré [42] indicate that, on general Lipschitz domains, the Sobolev smoothness of the solutions to (1) is given by $s^* = 1 + 1/p$ if $2 \leq p < \infty$, and by $s^* = 3/2$ if $1 < p < 2$. However, under certain conditions, the solutions possess higher regularity away from the boundary, in the sense that they are locally Hölder