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Stable estimates for source solution of critical fractal Burgers equation

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ABSTRACT

In this paper, we provide two-sided estimates for the source solution of d-dimensional critical fractal Burgers equation $u_t-\Delta^{\alpha/2}+b\cdot\nabla(u|u|^q)=0$, $q=(\alpha-1)/d$, $\alpha\in(1,2)$, $b\in\mathbb{R}^d$, by the density function of the isotropic α-stable process.

1. Introduction

Let $d\in\mathbb{N}$ and $\alpha\in(1,2)$. We consider the following pseudo-differential equation

$$\begin{cases} u_t-\Delta^{\alpha/2}u+b\cdot\nabla\left(u|u|^q\right)=0, & t>0,\ x\in\mathbb{R}^d,\\ u(0,x)=M\delta_0(x), \end{cases} \tag{1.1}$$

where $M>0$ is arbitrary constant and $b\in\mathbb{R}^d$ is a constant vector. In this paper, we focus on the critical case $q=(\alpha-1)/d$. Here, $\Delta^{\alpha/2}$ denotes the fractional Laplacian defined by the Fourier transform

$$\widehat{\Delta^{\alpha/2}\phi}(\xi)=-|\xi|^\alpha\widehat{\phi}(\xi),\quad \phi\in C_c^\infty(\mathbb{R}^d).$$

Eq. (1.1) for various values of q and initial conditions u_0 was recently intensely studied [2,4,3,7]. For $d=1$, the case $q=2$ is of particular interest (see e.g. [14,1,15,19]) because it is a natural counterpart of the classical Burgers equation. Another interesting value of q is $\frac{\alpha-1}{d}$. In [4] authors proved that the solution of (1.1), which we denote throughout the paper by $u_M(t,x)$, exists and is unique and positive. It belongs also to $L^p\left(\mathbb{R}^d\right)$ for every $p\in[1,\infty]$. The exponent $q=\frac{\alpha-1}{d}$ is critical in some sense. It is the only value for which the function $u_M(t,x)$ is self-similar. It satisfies the following scaling condition [4]

$$u_M(t,x)=a^d u_M(a^\alpha t,ax),\quad \text{for all } a>0. \tag{1.2}$$

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Furthermore, the linear and nonlinear terms in (1.1) have equivalent influence on the asymptotic behavior of the solution. If $q > (\alpha-1)/d$, the operator $\Delta^{\alpha/2}$ plays the main role. More precisely, for such q and a function u satisfying (1.1), with not necessarily the same initial condition, we have

$$\lim_{t\to\infty} t^{n(1-1/p)/\alpha} \left\| u(t,\cdot) - e^{\Delta^{\alpha/2}} u(0,\cdot) \right\|_p = 0, \quad \text{for each } p \in [1,\infty].$$

For $q < (\alpha-1)/d$ another asymptotic behavior is expected. In addition, taking $q = \frac{\alpha-1}{d}$ for $d = 1$ and $\alpha = 2$ we obtain the classical case, which makes Eq. (1.1) with critical exponent q one of the natural generalizations of the Burgers equation.

Till the end of the paper we assume that $d \geq 1, \alpha \in (1,2)$ and $q = \frac{\alpha-1}{d}$. Let $p(t,x)$ be the fundamental solution of

$$v_t = \Delta^{\alpha/2} v. \tag{1.3}$$

In [7] the authors proved that for sufficiently small M there is a constant $C = C(d,\alpha,M,b)$ such that

$$u_M(t,x) \leq Cp(t,x), \quad t > 0,\ x \in \mathbb{R}^d. \tag{1.4}$$

In this paper we get rid of the smallness assumption of M. Furthermore, we also obtain the lower bounds of u_M. We propose a new method which allows us to show pointwise estimates of solutions to the nonlinear problem (1.1) without the smallness assumption imposed on M. This method has been inspired by the proof of [6, Theorem 1]. Our main result is

Theorem 1.1. *Let $d \geq 1$ and $\alpha \in (1,2)$. Let $u_M(t,x)$ be the solution of Eq. (1.1) with $q = \frac{\alpha-1}{d}$. There exists a constant $C = C(d,\alpha,M,b)$ such that*

$$C^{-1}p(t,x) \leq u_M(t,x) \leq Cp(t,x), \quad t > 0,\ x \in \mathbb{R}^d.$$

In addition, applying Theorem 1.1, we get the following estimates

$$|u_M(1,x) - Mp(1,x)| \leq c\,\frac{p(1,x)}{1+|x|},$$

$$|\nabla u_M(t,x)| \leq c\,\frac{p(1,x)}{t^{-1/\alpha}+|x|}.$$

The fractional Laplacian plays also a very important role in the probability theory as a generator of the so called isotropic stable process. The theory of its linear perturbations has been recently significantly developed, see e.g., [5,6,12,13,18,16,17,8,10]. However, since the term $b \cdot \nabla(|u|^q u)$ in (1.1) represents a nonlinear drift, methods used in the linear case often cannot be adapted. In the proofs we mostly use the Duhamel formula and its suitable iteration. The scaling condition (1.2) is also intensively exploit.

The paper is organized as follows. In Preliminaries we collect some properties of the function $p(t,x)$ and introduce the Duhamel formula as well. In Section 3 we prove Theorem 1.1. In Section 4 we apply the methods and the results of Section 3 to obtain estimates of $|u_M(1,x) - Mp(1,x)|$ and $|\nabla u_M(1,x)|$.

2. Preliminaries

2.1. Notation

For two positive functions f, g we denote $f \lesssim g$ whenever there exists a constant $c > 1$ such that $f(x) < cg(x)$ for every argument x. If $f \lesssim g$ and $g \lesssim f$ we write $f \approx g$. If value of a constant in estimates is relevant, we denote it by $C_k, k \in \mathbb{N}$, and it does not change throughout the paper.