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Regularity of the free boundary for two-phase problems governed by divergence form equations and applications

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ABSTRACT

We study a class of two-phase inhomogeneous free boundary problems governed by elliptic equations in divergence form. In particular we prove that Lipschitz or flat free boundaries are $C^{1,\gamma}$. Our results apply to the classical Prandtl–Bachelor model in fluiddynamics.

1. Introduction and statements of the main theorems

This paper is a further step in the development of the theory for general elliptic inhomogeneous two-phase free boundary problems, after [8–10]. In particular, in [10], via Perron's method, we constructed a Lipschitz viscosity solution to problems governed by elliptic equations in divergence form with Hölder continuous coefficients and we proved weak measure theoretical regularity properties, such as "flatness" of the free boundary in a neighborhood of each point of its reduced part. Here, as in [8,9] we prove that flat or Lipschitz free boundaries are locally $C^{1,\gamma}$. It is worthwhile to notice that, in the absence of distributed sources and with Lipschitz coefficients, these regularity results were obtained in [13,14], while they are new even in the homogeneous case when the coefficients are assumed to be merely Hölder continuous.

Our setting is the following. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $A = \{a_{ij}(x)\}_{1\le i,j\le n}$ be a symmetric matrix with Hölder continuous coefficients in Ω, $A \in C^{0,\bar{\gamma}}(\Omega)$, which is uniformly elliptic, i.e.

$$\lambda \mid \xi \mid^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda \mid \xi \mid^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n$$

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for some $0 < \lambda \leq \Lambda$. Denote

$$\mathcal{L} := \operatorname{div}(A(x)\nabla\cdot).$$

Let $f \in L^\infty(\Omega)$. We consider the two-phase inhomogeneous free boundary problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega^+(u) = \{u > 0\} \\ \mathcal{L}u = f & \text{in } \Omega^-(u) = \{u \leq 0\}^\circ \\ |\nabla_A u^+|^2 - |\nabla_A u^-|^2 = 1 & \text{on } F(u) = \partial\{u > 0\} \cap \Omega, \end{cases} \tag{1.1}$$

where $|\nabla_A u|^2 := \langle A\nabla u, \nabla u\rangle$.

Since our emphasis is on the class of operators, we decided to avoid further technicalities by considering only a particular, although significant, free boundary condition. The extension to a general free boundary condition of the type $|\nabla u^+| = G(|\nabla u^-|, \nu, x)$, where $\nu = \nu(x)$ denotes the unit normal to $F(u)$ at x pointing towards $\Omega^+(u)$, can be achieved without much difficulty as in [8], if $G(\beta, x, \nu)$ is strictly increasing in β, Lipschitz continuous in the first and in the third argument, Hölder continuous in the second argument, $G(0) := \inf_{x\in\Omega, |\nu|=1} G(0, x, \nu) > 0$, and moreover $\eta^{-N} G(\eta, x, \nu)$ is strictly decreasing in η uniformly in x, ν.

We now recall the notion of viscosity solution. Here we give it in terms of test functions. In the last section we will use an equivalent notion in terms of asymptotic developments at one side regular points of the free boundary.

Definition 1.1. Given $u, \varphi \in C(\Omega)$, we say that φ touches u by below (resp. above) at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$, and

$$u(x) \geq \varphi(x) \quad (\text{resp. } u(x) \leq \varphi(x)) \text{ in a neighborhood } O \text{ of } x_0.$$

If this inequality is strict in $O \setminus \{x_0\}$, we say that φ touches u strictly by below (resp. above).

Definition 1.2. Let u be a continuous function in Ω. We say that u is a viscosity solution to (1.1) in Ω, if the following conditions are satisfied:

(i) $\mathcal{L}u = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the weak sense;
(ii) Let $x_0 \in F(u)$ and $v \in C^{1,\bar\gamma}(\overline{B^+(v)}) \cap C^{1,\bar\gamma}(\overline{B^-(v)})$ ($B = B_\delta(x_0)$) with $F(v) \in C^2$. If v touches u by below (resp.above) at $x_0 \in F(v)$, then

$$|\nabla_A v^+(x_0)|^2 - |\nabla_A v^-(x_0)|^2 \leq 1 \quad (\text{resp. } \geq).$$

We also need the definition of comparison subsolution (resp. supersolution).

Definition 1.3. We say that $v \in C(\Omega)$ is a $C^{1,\bar\gamma}$ strict (comparison) subsolution (resp. supersolution) to (1.1) in Ω, if $v \in C^{1,\bar\gamma}(\overline{\Omega^+(v)}) \cap C^{1,\bar\gamma}(\overline{\Omega^-(v)})$, $F(v) \in C^2$, and the following conditions are satisfied:

(i) $\mathcal{L}v > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$ in the weak sense;
(ii) If $x_0 \in F(v)$, then

$$|\nabla_A v^+(x_0)|^2 - |\nabla_A v^-(x_0)|^2 > 1 \quad (\text{resp. } |\nabla_A v^+(x_0)|^2 - |\nabla_A v^-(x_0)|^2 < 1.)$$

We notice that, using the almost monotonicity formula in [16], one can reproduce the proof of Theorem 4.5 in [5] to prove that viscosity solutions to (1.1) are locally Lipschitz continuous.

Our main Theorem is a “flatness implies regularity” result. Here, a constant depending (possibly) on $n, Lip(u), \lambda, \Lambda, [a_{ij}]_{C^{0,\bar\gamma}}, \|f\|_{L^\infty}$, is called universal.