Nonlinear Analysis 138 (2016) 83–92

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Weak convergence and averaging for ODE

Lawrence C. Evans*,[1], Te Zhang

Department of Mathematics, University of California, Berkeley, United States

ARTICLE INFO

Article history:
Received 3 June 2015
Accepted 8 October 2015
Communicated by Enzo Mitidieri

For Juan Luis Vazquez on his 70th birthday

Keywords:
Weak convergence
Stabilization
Adiabatic invariance

ABSTRACT

This mostly expository paper shows how weak convergence methods provide simple, elegant proofs of (i) the stabilization of an inverted pendulum under fast vertical oscillations, (ii) the existence of particle traps induced by rapidly varying electric fields and (iii) the adiabatic invariance of $\int_{\Gamma} p\,dx$ for slowing varying planar Hamiltonian dynamics. Under an appropriate, but very restrictive, unique ergodicity assumption, the proof of (iii) extends also to many degrees of freedom.

1. Introduction

The rigorous mathematical analysis of nonlinear differential equations depends primarily upon deriving estimates, but typically also upon using these estimates to justify limiting procedures of various sorts. For the latter, so-called weak convergence methods can be extremely valuable, as illustrated by many examples in the booklet [7].

This paper provides some more examples, concerning averaging effects for singularly perturbed nonlinear ODE. Section 2 shows how some simple “nonlinear resonance” effects (occurring when the weak limit of the product of two sequences of functions is not the product of the individual weak limits) appear for Kapitsa’s inverted pendulum and its generalizations. Section 3 invokes the more sophisticated tools of Young measures to document the adiabatic invariance of the volume within constant energy surfaces for slowly changing Hamiltonian systems, provided an appropriate ergodic type condition holds. Our proofs are perhaps new, at least in the elegant versions we provide, and our presentation is largely expository.

We wish also to call attention to Bornemann’s book [3], a very interesting discussion of weak convergence methods applied to singularly perturbed mechanical and quantum systems. His primary interest is explaining how increasingly singular potentials enforce holonomic constraints in the limit.

* Corresponding author.
E-mail address: evans@math.berkeley.edu (L.C. Evans).

[1] Class of 1961 Collegium Chair.

http://dx.doi.org/10.1016/j.na.2015.10.011

The results in Section 2 appear in somewhat different form in the second author's 2014 PhD thesis from UC Berkeley. We thank M. Zworski for explaining to us about ergodicity for Hamiltonian systems.

2. Averaging and stability

2.1. The inverted pendulum

The equation of motion for an inverted pendulum over a vertically oscillating pivot is

$$\theta_\epsilon'' - \left(a + \frac{b}{\epsilon}\cos\frac{t}{\epsilon}\right)\sin\theta_\epsilon = 0, \tag{2.1}$$

where $\theta_\epsilon = \theta_\epsilon(t)$ denotes the angle from the vertical and $a := \frac{g}{l} > 0$, l denoting the length. This is Kapitsa's pendulum: see for example Landau–Lifshitz [8, Section 30], Arnold [1, Section 25.E] and Levi [9].

We provide a simple proof that solutions of (2.1) converge as $\epsilon \to 0$ to solutions of $\theta'' + \frac{b^2}{4}\sin 2\theta - a\sin\theta = 0$. This ODE has the form $\theta_{tt} + F'(\theta) = 0$, for which the solution $\theta \equiv 0$ is stable provided $F''(0) = \frac{b^2}{2} - a > 0$; that is, if and only if $|b| \geq \sqrt{2a}$. This is the well-known stability condition for the inverted pendulum in the high frequency limit.

We turn now to a rigorous proof. Consider the following initial-value problem:

$$\begin{cases} \theta_\epsilon'' &= \left(a + \frac{b}{\epsilon}\cos\frac{t}{\epsilon}\right)\sin\theta_\epsilon \quad (t \geq 0) \\ \theta_\epsilon(0) &= \alpha \\ \theta_\epsilon'(0) &= \beta. \end{cases} \tag{2.2}$$

Theorem 2.1. *As $\epsilon \to 0$, θ_ϵ converges uniformly on each finite time interval $[0, T]$ to the solution θ of*

$$\begin{cases} \theta'' &= a\sin\theta - \frac{b^2}{4}\sin 2\theta \\ \theta(0) &= \alpha \\ \theta'(0) &= \beta. \end{cases} \tag{2.3}$$

The main idea will be to rewrite the ODE (2.2) into the form

$$\left(\theta_\epsilon' - b\sin\frac{t}{\epsilon}\sin\theta_\epsilon\right)' = a\sin\theta_\epsilon - b\sin\frac{t}{\epsilon}\cos\theta_\epsilon\theta_\epsilon'. \tag{2.4}$$

Proof. 1. First we show that for each $T > 0$, we have the estimate

$$\max_{0\leq t\leq T} |\theta_\epsilon|, |\theta_\epsilon'| \leq C_T, \tag{2.5}$$

for a constant $C_T > 0$ that only depends on T, α and β. To confirm this, integrate (2.4), to find

$$|\theta_\epsilon'(t)| \leq C_1 + C_2\int_0^t |\theta_\epsilon'|\, ds$$

for $0 \leq t \leq T$ and constants $C_1, C_2 \geq 0$. According then to Gronwall's inequality, we have the estimate

$$|\theta_\epsilon'(t)| \leq C_1\left(1 + C_2 t e^{C_1 t}\right) \leq C_T$$

for each $0 \leq t \leq T$ and a constant $C_T > 0$ that only depends on T.