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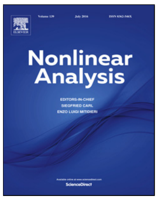

Notes on a PDE system for biological network formation

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ABSTRACT

We present new analytical and numerical results for the elliptic–parabolic system of partial differential equations proposed by Hu and Cai, which models the formation of biological transport networks. The model describes the pressure field using a Darcy's type equation and the dynamics of the conductance network under pressure force effects. Randomness in the material structure is represented by a linear diffusion term and conductance relaxation by an algebraic decay term. The analytical part extends the results of Haskovec et al. (2015) regarding the existence of weak and mild solutions to the whole range of meaningful relaxation exponents. Moreover, we prove finite time extinction or break-down of solutions in the spatially one-dimensional setting for certain ranges of the relaxation exponent. We also construct stationary solutions for the case of vanishing diffusion and critical value of the relaxation exponent, using a variational formulation and a penalty method.

The analytical part is complemented by extensive numerical simulations. We propose a discretization based on mixed finite elements and study the qualitative properties of network structures for various parameter values. Furthermore, we indicate numerically that some analytical results proved for the spatially one-dimensional setting are likely to be valid also in several space dimensions.

1. Introduction

In [9] we presented a mathematical analysis of the PDE system modeling formation of biological transportation networks

$$-\nabla \cdot [(rI + m \otimes m)\nabla p] = S, \tag{1.1}$$

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$$\frac{\partial m}{\partial t} - D^2 \Delta m - c^2 (m \cdot \nabla p)\nabla p + \alpha |m|^{2(\gamma-1)} m = 0, \tag{1.2}$$

for the scalar pressure $p = p(t,x) \in \mathbb{R}$ of the fluid transported within the network and vector-valued conductance $m = m(t,x) \in \mathbb{R}^d$ with $d \le 3$ the space dimension. The parameters are $D \ge 0$ (diffusivity), $c > 0$ (activation parameter), $\alpha > 0$ and $\gamma \in \mathbb{R}$; in particular, we restricted ourselves to $\gamma \ge 1$ in [9]. The scalar function $r = r(x) \ge r_0 > 0$ describes the isotropic background permeability of the medium. The term $S = S(x)$ models time-independent sources and sinks. The system was originally derived in [10,11] as the formal gradient flow of the continuous version of a cost functional describing formation of biological transportation networks on discrete graphs. In this context, (1.1) can be interpreted as Kirchhoff's law for the flux $u := -(rI + m \otimes m)\nabla p$, and the cost is proportional to $|u \cdot \nabla p| + \mathrm{const}|m|^{2\gamma}$. Therefore, the parameter $\gamma \in \mathbb{R}$ is crucial for the type of networks formed. For instance, modeling blood flow by Hagen–Poiseuille's law, $|m|$ is proportional to the square of the luminal diameter of the blood vessel. Furthermore, the metabolic cost for a blood vessel, which is described by $|m|^{2\gamma}$ in our notation, is proportional to the cross-sectional area of the blood vessel [14]. Therefore, for blood vessel systems, $\gamma = 1/2$. For leaf venation the situation is different since transportation of water causes no costs as it is driven by transpiration, which is due to external solar power [18]. However, the vessel structure influences photosynthesis, and, therefore, veins will build in such a way that the surface area between xylem water and surrounding mesophyll is maximized [20], which leads again to $\gamma = 1/2$. Since the mechanical stability of a leaf may depend on the structure of the vessels [18], also larger values of $\gamma > 1/2$ are likely. As argued in [11], if the metabolic costs are proportional to the number of conduits in a tube, then $\gamma = 1$. For further details on the modeling we refer to [1]. Related approaches may be found for instance in [5,7,12,23].

We pose (1.1), (1.2) on a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$, and, for simplicity, prescribe the homogeneous Dirichlet boundary conditions on $\partial\Omega$ for m and p:

$$m(t,x) = 0, \qquad p(t,x) = 0 \quad \text{for } x \in \partial\Omega,\ t \ge 0. \tag{1.3}$$

We note that the analytical results presented below can be easily extended to the case of homogeneous Neumann boundary conditions. Moreover, we prescribe the initial condition for m:

$$m(t=0,x) = m^0(x) \quad \text{for } x \in \Omega. \tag{1.4}$$

The main mathematical interest of the PDE system for network formation stems from the highly unusual nonlocal coupling of the elliptic equation (1.1) for the pressure p to the reaction–diffusion equation (1.2) for the conductance vector m via the pumping term $+c^2(\nabla p \otimes \nabla p)m$ and the latter term's potential equilibration with the decay term $-|m|^{2(\gamma-1)}m$. A major observation concerning system (1.1)–(1.2) is that it represents the formal $L^2(\Omega)$-gradient flow associated with the highly non-convex energy-type functional

$$\mathcal{E}(m) := \frac{1}{2}\int_\Omega \left(D^2 |\nabla m|^2 + \frac{\alpha}{\gamma}|m|^{2\gamma} + c^2 |m \cdot \nabla p[m]|^2 + c^2 r(x) |\nabla p[m]|^2 \right) \mathrm{d}x, \tag{1.5}$$

where $p = p[m] \in H_0^1(\Omega)$ is the unique solution of the Poisson equation (1.1) with given m, subject to the homogeneous Dirichlet boundary condition on $\partial\Omega$. Note that (1.5) consists of, respectively, the diffusive energy term, metabolic (relaxation) energy, and the last two terms account for network-fluid interaction energy. We have:

Lemma 1 (*Lemma 1 in [9]*). *Let $\mathcal{E}(m^0) < \infty$. Then the energy $\mathcal{E}(m(t))$ is nonincreasing along smooth solutions of (1.1)–(1.2) and satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(m(t)) = -\int_\Omega \left(\frac{\partial m}{\partial t}(t,x) \right)^2 \mathrm{d}x.$$