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Dirichlet conditions at infinity for parabolic and elliptic equations

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ABSTRACT

We investigate existence and uniqueness of bounded solutions of parabolic equations with unbounded coefficients in $\mathbb{R}^N \times \mathbb{R}_+$. Under specific assumptions, we establish existence of solutions satisfying prescribed conditions at infinity, depending on the direction along which infinity is approached. Moreover, the large-time behavior of such solutions is addressed. We consider also elliptic equations in $\mathbb{R}^N$ with similar conditions at infinity.

1. Introduction

We are concerned with bounded solutions of linear parabolic Cauchy problems of the following form:

$$\begin{cases} \partial_t u = \mathcal{L}u + cu + f & \text{in } S := \mathbb{R}^N \times (0, \infty), \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.1}$$

where $c, f, u_0 \in L^\infty(\mathbb{R}^N), N \geq 2$, and $\mathcal{L}$ is a second-order elliptic operator with possibly unbounded coefficients, defined by

$$\mathcal{L}u := \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i}.$$

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Moreover, we address bounded solutions of linear elliptic equations of the type

$$-\mathcal{L}u - cu = f \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

with $c \leq 0$. Precise assumptions on $\mathcal{L}, c, f$, and u_0 will be made in Section 2. We will also discuss some existence results for the *nonlinear* parabolic problem

$$\begin{cases} \partial_t u = \mathcal{L}[G(u)] + f & \text{in } S_T := \mathbb{R}^N \times (0,T), \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}. \end{cases} \tag{1.3}$$

Here $G(u)$ is a suitable function; in particular, we can choose $G(u) = |u|^{m-1}u$, which describes a *porous medium* type nonlinear diffusion. Also, uniqueness is established in the special case where problem (1.3) is of the form

$$\begin{cases} \rho\partial_t u = \mathcal{M}[G(u)] + f & \text{in } S_T, \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.4}$$

where $\rho \in C(\mathbb{R}^N), \rho > 0$, and $\mathcal{M}$ is a second-order elliptic operator in divergence form, formally defined by

$$\mathcal{M}v := \operatorname{div}\{A(x)\nabla v\}. \tag{1.5}$$

Suppose that the coefficients a_{ij}, b_i, c and the function f are bounded in $\mathbb{R}^N$. Then the Cauchy problem (1.1) is well posed in the class of bounded solutions. It is also well posed in the class of bounded solutions, if the coefficients grow as $|x| \to \infty$, but not too fast (see [13]). On the other hand, if the coefficients grow too fast, then the solution may be not unique (see [13,25]). The question that naturally arises then is which conditions may be prescribed at infinity in order to restore the well-posedness of the Cauchy problem. This question has been subject to detailed investigations (see, e.g., [9,10,12,14–17,22,23,27]). In particular, let us mention that, by the results in [17], for any given $a \in C([0,\infty))$ there exists a bounded solution to problem (1.1) (with $c \equiv 0, f \equiv 0$) such that

$$\text{for each } T > 0,\ \tau \in (0,T), \quad \lim_{|x|\to\infty} u(x,t) = a(t) \quad \text{uniformly with respect to } t \in [\tau, T], \tag{1.6}$$

provided there exists a supersolution $W = W(x) > 0$ to the equation

$$-\mathcal{L}W = 1 \quad \text{in } \mathbb{R}^N \setminus \overline{B_R}, \tag{1.7}$$

for some $R > 0$, such that

$$\lim_{|x|\to\infty} W(x) = 0. \tag{1.8}$$

Obviously, whether such a supersolution $W(x)$ exists or not depends on the coefficients of the operator $\mathcal{L}$ and c. For instance, if $N \geq 3$,

$$\mathcal{L}u = \frac{1}{\rho(x)}\Delta u$$

and

$$\rho(x) \leq C_0(1+|x|)^{-\alpha} \quad \text{for all } x \in \mathbb{R}^N, \quad \text{for some constants } C_0 > 0,\ \alpha > 2, \tag{1.9}$$

then $W := \Gamma * \rho$ satisfies (1.7), (1.8). Here Γ is the fundamental solution of the Laplace equation (see [5, 12]; see also [24]).

On the other hand, if the coefficients of $\mathcal{L}$ and c are bounded or grow slowly at infinity, then such $W(x)$ cannot exist. Condition (1.6) can be regarded as a condition at infinity for solutions to the Cauchy problem (1.1). The uniqueness of bounded solutions satisfying (1.6) is shown under some extra conditions. Note that