Nonlinear Analysis 138 (2016) 176–188

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Boundary behaviour for a singular perturbation problem

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ARTICLE INFO

Article history:
Received 3 August 2015
Accepted 23 December 2015
Communicated by Enzo Mitidieri

To Juan-Luis Vazquez on the occasion of his 70th Birthdate

MSC:
primary 35R35

Keywords:
Free boundary problem
Regularity
Contact points

ABSTRACT

In this paper we study the boundary behaviour of the family of solutions $\{u^\varepsilon\}$ to singular perturbation problem $\Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon), |\ u^\varepsilon\ |\leq 1$ in $B_1^+ = \{x_n > 0\}\cap\{|\ x\ |< 1\}$, where a smooth boundary data f is prescribed on the flat portion of ∂B_1^+. Here $\beta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\beta\left(\frac{\cdot}{\varepsilon}\right), \beta\in C_0^\infty(0,1), \beta\geq 0, \int_0^1\beta(t) = M > 0$ is an approximation of identity. If $\nabla f(z) = 0$ whenever $f(z) = 0$ then the level sets $\partial\{u^\varepsilon > 0\}$ approach the fixed boundary in tangential fashion with uniform speed. The methods we employ here use delicate analysis of local solutions, along with elaborated version of the so-called monotonicity formulas and classification of global profiles.

1. Introduction

In this paper we study the boundary behaviour of the family of solutions $\{u^\varepsilon\}$ to singular perturbation problem

$$\begin{cases} \Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon), & \text{in } B_1^+, \\ u^\varepsilon = f, & \text{on } B_1', \\ |u^\varepsilon|\leq 1, & \text{in } B_1^+, \end{cases} \tag{1.1}$$

in the half unit ball $B_1^+ = \{x_n > 0\}\cap\{|x| < 1\}$, where $x = (x_1,\ldots,x_n)\in\mathbb{R}^n_+$, and $B_1' = B_1\cap\{x_n = 0\}$. The perturbed right hand side β_ε, satisfies certain conditions that are specified below. Also, the boundary data f is a smooth function satisfying the following condition (specially on the flat portion of the boundary)

$$\nabla f(z) = 0 \quad \text{whenever } f(z) = 0. \tag{1.2}$$

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http://dx.doi.org/10.1016/j.na.2015.12.024
0362-546X/

Under these conditions we show that close to a “touching” point between the free and the fixed boundary, the free boundary touches the fixed one in a uniformly tangential fashion. Here free boundary refers to the zero level surface of our solution, $\partial\{u^\varepsilon > 0\}$.

Our analysis is based on utilization of the monotonicity formula and classification of global/blow-up solutions. The analogous problem for minimizers of the functional

$$J(u) = \int_{B_1^+} |\nabla u|^2 + \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u\le 0\}}$$

is studied in [8], where $\lambda_+^2 - \lambda_-^2 > 0$.

Problem (1.1) appears in the mathematical theory of combustion as a model with high activation energy, which is of order $\frac{1}{\varepsilon}$, in an ε-strip approximation of the flame, see [10, Chapter 4.3]. The family $\{\beta_\varepsilon(\cdot)\}$ renders such approximation (see (1.3)). Also, for more recent mathematical treatment see [2,4,5] and references therein.

Problem set-up and Standing Assumption:

To fix the ideas we suppose that

$$\beta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\beta\left(\frac{\cdot}{\varepsilon}\right), \quad \beta \in C_0^\infty(0,1),\ \beta \ge 0, \qquad \int_0^1 \beta(t)dt = M > 0. \tag{1.3}$$

Observe that by definition of $\beta_\varepsilon(t)$ we have

$$\int_0^\varepsilon \beta_\varepsilon(t)dt = \int_0^1 \beta(t)dt = M > 0.$$

The limit function, obtained as $\varepsilon \to 0$ solves locally the following free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cup \{u < 0\}, \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 2M & \text{on } \partial\{u > 0\}. \end{cases} \tag{1.4}$$

in a very weak sense, see [2,4,5].

Let f be a smooth function on $\{x_n = 0\} \cap B_1$ such that (1.2) is satisfied. It is known that under (1.2) the family $\{u^\varepsilon\}$ is uniformly bounded in Lipschitz norm [7]

$$\sup_{x \in B_{1/2}^+} |\nabla u^\varepsilon(x)| \le L, \tag{1.5}$$

with a positive constant $L > 0$, which is independent of ε for any solution of (1.1).

Assumptions (1.3) are standard (see [2,7]), however one can relax the assumption $\beta \in C_0^\infty(0,1)$ to $\beta \in C_0^{0,1}(0,1)$ in the proof of the Lipschitz norm estimate (1.5).

Non-degeneracy: Throughout the paper we shall assume a linear non-degeneracy at the origin, standard for such problems, which is

$$\int_{B_r^+} u \ge C_0 r^{n+1}, \tag{1.6}$$

for a universal C_0.

Remark 1.1. If large enough negative and positive phases are present then one can prove that u^+ is non-degenerate. Namely, let $x_0 \in \partial\{u > 0\}$, if there is a unit vector e, such that

$$\liminf_{r\to 0} \frac{|\{u > 0\} \cap \{(x - x_0)\cdot e > 0\} \cap B_r(x_0)|}{|B_r|} = \alpha_1 \tag{1.7}$$
$$\liminf_{r\to 0} \frac{|\{u < 0\} \cap \{(x - x_0)\cdot e < 0\} \cap B_r(x_0)|}{|B_r|} = \alpha_2$$

with $\alpha_1 + \alpha_2 > \frac{1}{2}$ then there exists a tame constant $C > 0$ such that $\sup_{B_r(x_0)} u^\varepsilon \ge Cr$ [5, Theorem 6.3].