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Asymptotic analysis of a doubly nonlinear diffusion equation

J.R. King

School of Mathematical Sciences, University of Nottingham, NG7 2RD, United Kingdom

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ABSTRACT

We investigate the doubly nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{1}{n}\nabla.\left(u^m|\nabla u|^{n-1}\nabla u\right)$$

and the same equation expressed in terms of a 'pressure' variable. We classify various classes of compacted supported solutions, as well as finite-mass solutions that decay algebraically at infinity. A number of novel phenomena are identified, particularly for $n < 0$, that seem to us worthy of further mathematical investigation.

1. Introduction

We are concerned in this paper with the doubly nonlinear equation

$$\frac{\partial u}{\partial t} = \frac{1}{n}\nabla \cdot (u^m|\nabla u|^{n-1}\nabla u), \tag{1}$$

which we shall also write without comment in the rescaled form

$$\frac{\partial u}{\partial t} = sgn(n)\nabla \cdot (u^m|\nabla u|^{n-1}\nabla u), \tag{2}$$

depending on context. As the notation already implies, we consider only solutions with $u \geq 0$ everywhere. We shall be mainly concerned with finite-mass solutions and shall seek to explore the full parameter space of the exponents (m, n). We have a number of distinct motivations. Firstly, (1) has two particularly well-studied special cases, namely the porous medium equation (PME)[1]

$$(n = 1) \quad \frac{\partial u}{\partial t} = \nabla \cdot (u^m\nabla u) \tag{3}$$

E-mail address: john.king@nottingham.ac.uk.
URL: http://www.nottingham.ac.uk/mathematics/index.aspx.

[1] We retain the PME and PLE terminology whatever the values (and in particular the signs) of the exponents m and n in (3) and (4).

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and the p-Laplace equation (PLE)

$$(m = 0) \quad \frac{\partial u}{\partial t} = \nabla \cdot (|\nabla u|^{n-1}\nabla u); \tag{4}$$

see Vázquez [20,19] for accounts of the extensive mathematical theory of these evolution equations. In one dimension there is an obvious relationship between (3) and (4), with $n = m + 1$, but – as we shall see – the two can have very different qualitative properties; (1) allows the transition between the two be explored. Secondly, (1) has a number of physical applications, though our focus will not be on these. Finally, and perhaps most significantly here, equations of a similar type to (1) have been adopted in image-analysis applications, motivating a detailed exploration of the range of qualitative properties that (1) can exhibit and we emphasis in advance that there are regimes of (m, n) parameter space in which phenomena not shared by (3) or (4) arise.

When $n < 0$ we shall restrict ourselves to the one-dimensional case for the following reason, in effect well known in the image-analysis context: adopting the local expansion

$$u \sim u_0 + u_1 x_1 + U,$$

(1) implies in N dimensions and at leading order that

$$\frac{\partial U}{\partial t} = u_0^m |u_1|^{n-1} \left(\frac{\partial^2 U}{\partial x_1^2} + \frac{1}{n} \left(\frac{\partial^2 U}{\partial x_2^2} + \cdots + \frac{\partial^2 U}{\partial x_N^2} \right) \right),$$

i.e. U satisfies for $n < 0$ a forward heat equation in the direction of the gradient of u but a backward one in directions orthogonal to that gradient. In addition, in one dimension only can more general formulations such as

$$\frac{\partial w}{\partial t} = \frac{1}{\mu} w^\alpha \left| \frac{\partial w}{\partial x} \right|^\beta \frac{\partial}{\partial x} \left(w^\gamma \left| \frac{\partial w}{\partial x} \right|^{\mu-1} \frac{\partial w}{\partial x} \right) \tag{5}$$

be reexpressed in the form (1): setting

$$w = \phi^{\mu/(\gamma+\mu)}$$

enables (5) to be written, up to rescaling, in the form

$$\frac{\partial \phi}{\partial t} = \frac{1}{n} \phi^\sigma \frac{\partial}{\partial x} \left(\left| \frac{\partial \phi}{\partial x} \right|^{n-1} \frac{\partial \phi}{\partial x} \right)$$

where $n = \mu + \beta, \sigma = (\gamma(1-\beta) + \alpha\mu)/(\gamma + \mu)$, and (1) follows on introducing $u = \phi^{1-\sigma}$.

The remainder of the paper is organised as follows. In Section 2 we outline a number of preliminary results that serve to guide the detailed analyses in Section 3 of $n > 0$ and in Section 4 of $n < 0$. Sections 3 and 4 each include a summary of the main results therein, so we conclude in Section 5 with a brief discussion. Appendix A notes some equivalence transformations that are of independent interest and can be exploited in mapping some of the IBVPs that we explore into ones studied in other contexts; we do not pursue this avenue here, however, preferring to focus on (1) as it stands, except in Section 4.4. Appendix B addresses the exceptional (borderline) case $n = 0$. Our analysis will be formal throughout and will sidestep some basic mathematical questions about the properties of the governing equations. We should remark at the outset that our discussion in what follows is in places rather speculative, and innumerable issues remain open (notably, but far from only, in higher dimensions), even at the formal level. We have attempted to keep the analysis fairly self-contained, and apologise for where we have failed to reference (or have inadequately referred to) existing literature.