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Nonlinear Potential Theory of elliptic systems

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ABSTRACT

We give a summary of recent results from Nonlinear Potential Theory, focusing on linear and nonlinear potential estimates of solutions to non-homogeneous equations and systems. We start with the cases of quasilinear, possibly degenerate equations of p-Laplacian type, and, passing through fully nonlinear elliptic equations, finally move to systems. In this last case we describe recent results implying potential estimates for the p-Laplacian system. Finally, we describe results bridging Nonlinear Potential Theory and classical partial regularity theory for general elliptic systems. The main outcomes are new ε-regularity criteria involving both excess functionals and nonlinear potentials.

1. Introduction and notation

The aim of this paper is to give a survey of recent developments in Nonlinear Potential Theory [52] that allow to extend the classical linear and nonlinear potential estimates, valid for scalar equations, to elliptic systems. As a consequence, several fine properties of solutions to nonlinear, potentially degenerate elliptic equations are discovered.

The plan of the paper is the following. In Section 2 we give a summary of a few by now classical results concerning single equations, i.e. we consider scalar solutions to measure data problems. The main emphasis is on linear and nonlinear potential estimates.

In Section 3 we give a sort of fully nonlinear interlude to show how the basic potential estimates for quasilinear, possibly degenerate equations shown in Section 2 admit a neat reformulation in the context of fully nonlinear elliptic equations. The main concept here is the one of modified Riesz potentials. These are actually nonlinear Wolff potentials scaling as Riesz potentials, and they replace classical Riesz potentials when these cannot be used. This is indeed the case of non-homogeneous fully nonlinear equations, where

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measure data are not allowed. This approach will come back later, in another situations where no theory of measure data is available; this is the case of general systems in the last section.

Then, in Section 4 we shall report on the more recent results obtained in [77]. Here we describe how exactly the same potential estimates available in the scalar case are indeed available in the case of the non-homogeneous p-Laplacian system (4.1). These estimates are very general and allow to reproduce in the vectorial case a whole bunch of regularity results for scalar measure data problems that were still missing in the vectorial case.

In Section 5 we show how to build a bridge between classical partial regularity theory of general elliptic systems and Nonlinear Potential Theory. At this point the modified Riesz potentials already used for fully nonlinear equations are employed again since for general systems it is not possible to take measures as data. The key idea is to formulate the classical ε-regularity leading to partial regularity with new criteria using potentials. The classical conditions using smallness of excess functionals are replaced by conditions naturally prescribing both the smallness of the excess and the one of the relevant modified Riesz potential. Sharp pointwise estimates then follow and in turn they imply new optimal partial regularity results in borderline spaces. These are the precise extensions of those valid for scalar equations.

We conclude this introductory section by briefly fixing some notation. Constants are generically denoted by c; these are larger than or equal to one; dependence on parameters is indicated using parenthesis. For instance, a constant c depending only on other quantities indicated by n, N, p, ν, L is denoted by $c \equiv c(n, N, p, \nu, L)$. In the following

$$B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

denotes the open ball with center x and radius $r > 0$. When not important, or it is clear from the context, we shall omit denoting the center as follows: $B_r \equiv B_r(x_0)$. With $\mathcal{O} \subset \mathbb{R}^n$ being a measurable subset with positive measure, and with $g\colon \mathcal{O} \to \mathbb{R}^k$, $k \geq 1$, being a measurable map, we shall denote by

$$(g)_{\mathcal{O}} \equiv \fint_{\mathcal{O}} g\,dx := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} g(x)\,dx$$

its integral average; here $|\mathcal{O}|$ denotes the Lebesgue measure of $\mathcal{O}$. In the following, μ will always denote a Borel measure with finite total mass, which is initially defined on a certain open subset $\Omega \subset \mathbb{R}^n$, and we shall always consider the multidimensional case $n \geq 2$. Since this will not affect the rest, with no loss of generality, all such measures will be considered as defined in the whole $\mathbb{R}^n$ so that

$$|\mu|(\mathbb{R}^n) < \infty.$$

2. Nonlinear potential theory

Here we consider elliptic equations with measure data of the type

$$-\operatorname{div} a(Du) = \mu \quad \text{in } \Omega. \tag{2.1}$$

We shall consider the following growth and ellipticity assumptions on the C^1-vector field $a(\cdot)$:

$$\begin{cases} |a(z)| + |\partial a(z)||z| \leq L|z|^{p-1} \\ \nu |z|^{p-2}|\xi|^2 \leq \langle \partial a(z)\xi, \xi\rangle \end{cases} \tag{2.2}$$

for every choice of $z, \xi \in \mathbb{R}^n$. Here $n \geq 2$, while $0 < \nu \leq L$ are fixed constants. These assumptions prescribe the differentiability of the vector field $a(\cdot)$ and can sometimes be weakened; the interested reader is referred to [74] for more general situations.