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Computing normal forms and formal invariants of dynamical systems by means of word series

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ABSTRACT

We show how to use extended word series in the reduction of continuous and discrete dynamical systems to normal form and in the computation of formal invariants of motion in Hamiltonian systems. The manipulations required involve complex numbers rather than vector fields or diffeomorphisms. More precisely we construct a group $\overline{\mathcal{G}}$ and a Lie algebra $\overline{\mathfrak{g}}$ in such a way that the elements of $\overline{\mathcal{G}}$ and $\overline{\mathfrak{g}}$ are families of complex numbers; the operations to be performed involve the multiplication $\bigstar$ in $\overline{\mathcal{G}}$ and the bracket of $\overline{\mathfrak{g}}$ and result in *universal* coefficients that are then applied to write the normal form or the invariants of motion of the specific problem under consideration.

1. Introduction

In this paper we show how to use extended word series in the reduction of continuous and discrete dynamical systems to normal form and in the computation of formal invariants of motion in Hamiltonian systems of differential equations. The manipulations required in our approach involve complex numbers rather than vector fields or diffeomorphisms. More precisely, we construct a group $\overline{\mathcal{G}}$ (semidirect product of the additive group of $\mathbb{C}^d$ and the group of characters of the shuffle Hopf algebra) and a Lie algebra $\overline{\mathfrak{g}}$ in such a way that the elements of $\overline{\mathcal{G}}$ and $\overline{\mathfrak{g}}$ are families of complex numbers; the operations to be performed involve the multiplication $\bigstar$ in $\overline{\mathcal{G}}$ and the bracket of $\overline{\mathfrak{g}}$ and result in *universal* coefficients that are then applied to write the normal form or the invariants of motion of the specific problem under consideration.[1]

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[1] In fact it is possible to present $\overline{\mathcal{G}}$ and $\overline{\mathfrak{g}}$ in terms of a *universal property* in the language of category theory. We shall not be concerned with that task here.

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The present approach originated from our earlier work on the use of formal series to analyze numerical integrators; see [15] for a survey of that use. In a seminal paper, Hairer and Wanner [11] introduced the concept of B-series as a means to perform systematically the manipulations required to investigate the accuracy of Runge–Kutta and related numerical methods for ordinary differential equations. B-series are series of functions; there is a term in the series associated with each rooted tree. The letter B here refers to John Butcher, who in the early 1960s enormously simplified, through a book-keeping system based on rooted trees, the task of Taylor expanding Runge–Kutta solutions. This task as performed by Kutta and others before John Butcher's approach was extraordinarily complicated and error prone. Key to the use of B-series is the fact that the substitution of a B-series in another B-series yields a third B-series, whose coefficients may be readily found by operating with the corresponding rooted trees and are independent of the differential equation being integrated. In this way the set of coefficients itself may be endowed with a product operation and becomes the so-called Butcher's group. The set of B-series resulting from a specific choice of differential equations is then a homomorphic image of the Butcher group. It was later discovered (see e.g. [4]) that the Butcher group was not a mere device to understand numerical integrators but an important mathematical construction (the group of characters of a Hopf algebra structure on the set of rooted trees) which has found applications in several fields, notably in renormalization theories and noncommutative geometry. In [5] and [6] B-series found yet another application outside numerical mathematics when they were employed to average systems of differential equation with periodic or quasiperiodic forcing.

Our work here is based on *word* series [13], an alternative to B-series defined in [7,8] (see also [12,6]). Word series are parameterized by the words of an alphabet A and are more compact than the corresponding B-series parameterized by rooted trees with colored nodes. Section 2 provides a review of the use of word series. The treatment there focuses on the presentation of the rules that apply when manipulating the series in practice and little attention is paid to the more algebraic aspects. In particular the material in Section 2 is narrowly related to standard constructions involving the shuffle Hopf algebra over the alphabet A (see [13] and its references); we avoid to make this connection explicit and prefer to give a self-contained elementary exposition. Section 3 presents the class of perturbed problems investigated in the paper; several subclasses are discussed in detail, including nonlinear perturbations of linear problems and perturbations of integrable problems written in action/angle variables [2]. In order to account for the format (unperturbed + perturbation) being considered, we employ what we call *extended word series*. In the particular case of action/angle integrable problems, extended word series have already been introduced in [13]; here we considerably enlarge their scope. Section 4, that is parallel to Section 2, studies the rules that apply to the manipulation of extended words series. Again many of those rules essentially appear in [13], but the *ad hoc* proofs we used there do not apply in our more general setting so that new, more geometric proofs are required. The main results of the paper are presented in the final Section 5. We first address the task of reducing differential systems to normal forms via changes of variables (Theorem 1); as pointed out above, both the normal form and the change of variables are written in terms of scalar coefficients that may be easily computed. For Hamiltonian problems the normal form is Hamiltonian and the change of variables symplectic. We also describe in detail (Theorem 2) the freedom that exists when finding the normal form/change of variables. By going back to the original variables, we find a decomposition of the given vector field as a commuting sum of two fields: the first generates a flow that is conjugate to the unperturbed problem and the second accounts for the effects of the perturbations that are not removable by conjugation. This decomposition is the key to the construction of formal invariants of motion in Hamiltonian problems. We provide very simple recursions for the computation of the coefficients that are required to write down the decomposition of the vector field and the invariants of motion. Finally we briefly outline a parallel theory for discrete dynamical systems.

Some of the results in Section 5 have precedents in the literature. The reduction to normal form (but not the investigation of the associated freedom) appears in [13] but only the particular setting of action/angle