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Single-point blow-up for parabolic systems with exponential nonlinearities and unequal diffusivities

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ABSTRACT

We study positive blowing-up solutions of systems of the form:

$$u_t = \delta_1 \Delta u + e^{pv}, \quad v_t = \delta_2 \Delta v + e^{qu},$$

with $\delta_1, \delta_2 > 0$ and $p, q > 0$. We prove single-point blow-up for large classes of radially decreasing solutions. This answers a question left open in a paper of Friedman and Giga (1987), where the result was obtained only for the equidiffusive case $\delta_1 = \delta_2$ and the proof depended crucially on this assumption.

1. Introduction

In this paper, we consider nonnegative solutions of the following reaction–diffusion system:

$$\begin{cases} u_t = \delta_1 \Delta u + f(v), & t > 0,\ x \in \Omega, \\ v_t = \delta_2 \Delta v + g(u), & t > 0,\ x \in \Omega, \end{cases} \tag{1.1}$$

with possibly unequal diffusivities $\delta_1, \delta_2 > 0$, and nonlinearities of exponential type, namely:

$$f(v) = e^{pv}, \quad g(u) = e^{qu}, \quad p,\ q > 0 \tag{1.2}$$

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or

$$f(v) = e^{pv} - 1, \quad g(u) = e^{qu} - 1, \quad p,\ q > 0. \tag{1.3}$$

Problem (1.1) is a basic model case for reaction–diffusion systems and, as such, it has been the subject of intensive investigation for more than 20 years (see e.g. [20, Chapter 32] and the references therein). We are here mainly interested in proving single-point blow-up for system (1.1) with exponential nonlinearities.

For system (1.1) and with $f,\ g$ given by (1.2), the blow-up set was first studied in [9]. In that work, Friedman and Giga considered symmetric nonincreasing solutions of the one-dimensional initial-Dirichlet problem and, under the restrictive condition $\delta_1 = \delta_2$, they proved that blow-up occurs only at the origin; see [9, Theorem 3.1, p. 73]. Note that the assumption $\delta_1 = \delta_2$ is essential in [9] in order to apply the maximum principle to suitable linear combinations of the components u and v, so as to derive comparison estimates between them. The problem for $\delta_1 \neq \delta_2$ was left open. In this instance, we recall that non-equidiffusive parabolic systems are often much more involved, both in terms of behavior of solutions and at the technical level (cf. [18] and [20, Chapter 33]).

The purpose of this paper is to give an answer to this question. We will actually consider more generally the radially symmetric problem in higher dimensions. In what follows, for $R \in (0, \infty]$, we denote $B_R = \{x \in \mathbb{R}^n\ ; |x| < R\}$, with $n \geq 1$ an integer (so, $B_R = \mathbb{R}^n$ for $R = \infty$). We say that (u, v) is radially symmetric nonincreasing if

$$\begin{aligned} &u = u(t, \rho), \qquad v = v(t, \rho) \quad \text{with } \rho = |x|, \\ &u_\rho,\ v_\rho \leq 0 \quad \text{for } 0 < t < T \text{ and } 0 < \rho < R. \end{aligned} \tag{1.4}$$

Our main result is the following.

Theorem 1.1 (*Single-Point Blow-up*)**.** *Let $\delta_1, \delta_2 > 0$, $T \in (0, \infty)$, $R \in (0, \infty]$ and $\Omega = B_R$. Let $f,\ g$ be given by (1.2) or (1.3). Let (u, v) be a nonnegative, radially symmetric nonincreasing, classical solution of (1.1) in $(0, T) \times \Omega$, with $u_\rho \not\equiv 0$ or $v_\rho \not\equiv 0$. Assume that (u, v) satisfies the type I blow-up estimates:*

$$q\|u(t)\|_{L^\infty(\Omega)} \leq |\log(T - t)| + C, \quad p\|v(t)\|_{L^\infty(\Omega)} \leq |\log(T - t)| + C, \quad 0 < t < T, \tag{1.5}$$

for some constant $C > 0$. Then blow-up occurs only at the origin, i.e.:

$$\sup_{0<t<T} (u(t,\ \rho) + v(t,\ \rho)) < \infty, \quad \textit{for all } \rho \in (0, R).$$

In order to produce actual solutions with single-point blow-up, we of course need to consider initial–boundary value problems associated with system (1.1), and in particular we have to ensure the type I blow-up assumption (1.5). This terminology of "type I blow-up", which comes from the theory of curvature flows in differential geometry (cf. Hamilton [12]), has now become standard in the field of nonlinear PDE's (cf., e.g., Matano–Merle [17]). It means that the blowup rate of the solution has an upper bound of same order as blowup solutions of the corresponding ODE problem. Blowup is said to be type II otherwise. For $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, we consider the Dirichlet problem

$$\begin{cases} u_t = \delta_1 \Delta u + f(v), & t > 0,\ x \in \Omega, \\ v_t = \delta_2 \Delta v + g(u), & t > 0,\ x \in \Omega, \\ u(t, x) = v(t, x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0, x) = u_0(x), \qquad v(0, x) = v_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$