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Pinning with a variable magnetic field of the two dimensional Ginzburg–Landau model

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A B S T R A C T

We study the Ginzburg–Landau energy of a superconductor with a variable magnetic field and a pinning term in a bounded smooth two dimensional domain Ω. Supposing that the Ginzburg–Landau parameter and the intensity of the magnetic field are large and of the same order, we determine an accurate asymptotic formula for the minimizing energy. This asymptotic formula displays the influence of the pinning term. Also, we discuss the existence of non-trivial solutions and prove some asymptotics of the third critical field.

1. Introduction

We consider a bounded, open and simply connected set $\Omega \subset \mathbb{R}^2$ with smooth boundary. We suppose that Ω models an inhomogeneous superconducting sample submitted to an applied external magnetic field. The energy of the sample is given by the so called pinned Ginzburg–Landau functional,

$$\mathcal{E}_{\kappa,H,a,B_0}(\psi,\mathbf{A}) = \int_\Omega \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(a(x,\kappa) - |\psi|^2)^2 \right) dx + \kappa^2 H^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2\, dx. \quad (1.1)$$

Here κ and H are two positive parameters such that κ describes the properties of the material, and H measures the variation of the intensity of the applied magnetic field. The modulus $|\psi|^2$ of the wave function (order parameter) $\psi \in H^1(\Omega;\mathbb{C})$ measures the density of the superconducting electron Cooper pairs. The

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magnetic potential $\mathbf{A}$ belongs to $H^1_{\mathrm{div}}(\Omega)$ where

$$H^1_{\mathrm{div}}(\Omega) = \{\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2) \in H^1(\Omega)^2 : \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega,\ \mathbf{A} \cdot \nu = 0 \text{ on } \partial\Omega\}, \tag{1.2}$$

with ν being the unit interior normal vector of $\partial\Omega$.

The function $\kappa H \operatorname{curl} \mathbf{A}$ gives the induced magnetic field.

When $\psi \equiv 0$ and $(\psi, \mathbf{A})$ is a minimizer or a critical point of the functional, we call this pair normal state. In our case it is easy to see normal minimizers (if any) are necessarily in the form $(0, \mathbf{A})$ with $\mathbf{A}$ in $H^1_{\mathrm{div}}(\Omega)$ such that $\operatorname{curl} \mathbf{A} = B_0$. This solution is unique and denoted by $\mathbf{F}$. A natural question will be to determine under which condition this normal solution is a minimizer.

The function $B_0 \in C^\infty(\overline{\Omega})$ is the intensity of the external magnetic field which is variable in our problem. Let

$$\Gamma = \{x \in \overline{\Omega} : B_0(x) = 0\}. \tag{1.3}$$

We assume that either Γ is empty or that B_0 satisfies:

$$\begin{cases} |B_0| + |\nabla B_0| > 0 & \text{in } \overline{\Omega} \\ \nabla B_0 \times \vec{n} \neq 0 & \text{on } \Gamma \cap \partial\Omega. \end{cases} \tag{1.4}$$

The assumption in (1.4) implies that for any open set ω relatively compact in Ω, $\Gamma \cap \omega$ is either empty, or consists of a union of smooth curves.

The energy $\mathcal{E}_{\kappa,H,a,B_0}$ considered here is slightly different from the classical Ginzburg–Landau energy in the sense that there is a varying term denoted by $a(x,\kappa)$ penalizing the variations of the order parameter ψ and called the pinning term. This term arises also naturally in the microscopic derivation of the Ginzburg–Landau theory from BCS theory (see [18]) without any a priori assumption on the sign of a.

In this paper, we will assume that the pining term a satisfies:

Assumption 1.1. The function $a(x,\kappa)$ is real, defined on $\overline{\Omega} \times [\kappa_0, +\infty)$, and satisfies for some $\kappa_0 > 0$ the following assumptions:

(A_1)

$$\forall \kappa \geq \kappa_0, a(\cdot, \kappa) \in C^1(\overline{\Omega}). \tag{1.5}$$

(A_2)

$$\sup_{x \in \overline{\Omega},\, \kappa \geq \kappa_0} |a(x,\kappa)| \leq 1. \tag{1.6}$$

(A_3)

$$\sup_{x \in \overline{\Omega},\, \kappa \geq \kappa_0} |\nabla_x a(x,\kappa)| < +\infty. \tag{1.7}$$

(A_4) There exists a positive constant C_1, such that,

$$\forall \kappa \geq \kappa_0, \qquad \mathcal{L}\left(\partial\{a(x,\kappa) > 0\}\right) \leq C_1 \kappa^{\frac{1}{2}}, \tag{1.8}$$

where $\mathcal{L}$ is the "length" of $\partial\{a(x,\kappa) > 0\}$ in Ω in a sense that will be explained in (3.1).