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Slope estimate and boundary differentiability of infinity harmonic functions on convex domains☆

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ABSTRACT

We study the boundary differentiability of infinity harmonic functions with given differentiable boundary data on convex domains. At a flat point (the boundary point where the blow-up of the domain is the half-space), the infinity harmonic function u is differentiable due to a previous result of the first author in Hong (2013). At a corner point (the boundary point where the blow-up of the domain is not the half-space), an example shows that u is not necessarily differentiable. In this paper, we establish a slope estimate for u at corner points and provide a necessary and sufficient condition for the differentiability of u at corner points.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a connected open set, an infinity harmonic function $u \in C(\Omega)$ is a viscosity solution of the infinity Laplace equation

$$\triangle_\infty u(x) := \sum_{1\le i,j\le n} u_{x_i} u_{x_j} u_{x_i x_j} = 0, \quad x \in \Omega.$$

The above equation was introduced by G. Aronsson in the 1960s [2,3] as the Euler–Lagrange equation of the *sup-norm variational problem* of $|\nabla u|$ or the equivalent optimal Lipschitz extension problem.

Bhattacharya et al. [5] proved the existence of infinity harmonic functions with a given boundary datum and Jensen [12] proved the uniqueness (see also [1]). Jensen [12] also showed that a function $u \in C(\Omega)$ is an infinity harmonic function if and only if u is an *absolutely minimizing Lipschitz extension* that means u

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satisfies the following property: for any open set $V \subset\subset \Omega$,

$$\sup_{x\neq y\in\partial V}\frac{|u(x)-u(y)|}{|x-y|}=\sup_{x\neq y\in\bar{V}}\frac{|u(x)-u(y)|}{|x-y|}.$$

In 2001, Crandall–Evans–Gariepy [7] introduced the revolutionary *comparison with cones property*. They proved that $u \in C(\Omega)$ is an infinity harmonic function if and only if u enjoys the comparison with cones property: for any $V \subset\subset \Omega$ and any cone function $C(x) = a + b|x - z|$ with $a, b \in R$,

$$\begin{aligned} u(x) \le C(x) \quad &\text{on } \partial(V\backslash\{z\}) \Rightarrow u(x) \le C(x) \quad \text{in } V; \\ u(x) \ge C(x) \quad &\text{on } \partial(V\backslash\{z\}) \Rightarrow u(x) \ge C(x) \quad \text{in } V. \end{aligned}$$

The interior differentiability of u was achieved by Evans–Smart [9]. For dimension 2, Savin [14] and Evans–Savin [8] proved the C^1 and $C^{1,\alpha}$ regularity earlier. The continuous interior differentiability of u for general dimensions remains the most important open problem in this field.

The boundary regularity of infinity harmonic functions was initially studied by Wang–Yu [15] and followed by the first author of this paper [10,11]. Wang–Yu proved that u is differentiable on the boundary if both $\partial\Omega$ and the boundary condition g are C^1. For dimension 2, they proved that u is C^1 on the boundary if both $\partial\Omega$ and the boundary condition g are C^2. In [10], we improved their first result to the following: if both $\partial\Omega$ and g are differentiable at a boundary point $x_0 \in \partial\Omega$, then u is differentiable at x_0. In [11], the author provided a counterexample to show that $|Du|$ can be discontinuous along the boundary if we only assume $\partial\Omega$ is C^1 even if g is smooth and the dimension is 2.

In this paper, we further study the boundary regularity by considering the convex domains. At a boundary point of a convex domain, the blow-up of the domain always uniquely exists and is a convex cone. There are exactly two cases: if the blow-up is a half-space, we call the boundary point a *flat point*, and in this case, $\partial\Omega$ is differentiable at this point; if the blow-up is not a half-space, we call the boundary point a *corner point*, and in this case, Ω is contained in the intersection of two different half-spaces. At a flat point, the boundary is differentiable, thus u is differentiable at this point if g is so due to the result in [10]. The corner point case is more complicated and interesting. Example 1 in Section 2 shows that the differentiability of g cannot guarantee the differentiability of u in general. In Section 3, we prove that if g is differentiable at a corner point x_0, then the slope function (defined in Section 2) $S(x_0) \le |Dg(x_0)|$. That is, we have good control on the slope of u at x_0 although u may be not differentiable at x_0. In Section 4, we prove that if g is differentiable at a corner point x_0, then when $x_0 + tDg(x_0) \notin \Omega$ for all $t \in \mathbb{R}u$ is differentiable at x_0; when $x_0 + tDg(x_0) \in \Omega$ for some $t \in \mathbb{R}u$ is differentiable at x_0 if and only if $S(x_0) = |Dg(x_0)|$.

It is very interesting to compare our result with the work of Li D.S. and Wang L.H. for uniformly elliptic equations on convex domains [13]. For uniformly elliptic equations, at corner points, the differentiability of the boundary data g guarantees the differentiability of the solution; while at flat points, an extra Dini condition on g is necessary for the differentiability of the solution (see Theorem 1.2 in [13]).

2. Preliminary

Throughout this paper, we assume $\Omega \subset \mathbb{R}^n$ is a bounded convex domain, and $u \in C(\bar{\Omega})$ is an infinity harmonic function in Ω and $u|_{\partial\Omega} = g$.

Let $x_0 \in \partial\Omega$ is a boundary point under study. The set $\Omega^t_{x_0} := \{t(y - x_0) : y \in \Omega\}$ with $t > 0$ is nondecreasing due to the convexity of Ω. So the blow up of Ω at $x_0\Omega^\infty_{x_0} := \bigcup_{t>0}\Omega^t_{x_0}$ exists and is a convex cone with 0 as its vertex. If $\Omega^\infty_{x_0} = \{x_n > 0\}$ under some coordinates system, we say x_0 is a flat point. Otherwise, $\Omega^\infty_{x_0}$ is strictly contained in a half space. In this case, one can prove (using separation theorem of convex sets) that there exists $\delta > 0$, such that $\Omega^\infty_{x_0} \subset \{x_n > \delta|x_{n-1}|\}$ under some coordinates system, and we call x_0 a corner point.

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