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Some overdetermined problems for the fractional Laplacian equation on the exterior domain and the annular domain

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ABSTRACT

In this paper, by the method of moving planes, we extend the Serrin-type overdetermined problem to the fractional Laplacian equation on the exterior set of a multiply connected domain and the annular domain.

1. Introduction

On the symmetry and related properties of solutions of second order elliptic equations, there have been many elegant results. For example, it is well known that in the paper [36], J. Serrin researched solutions of the following equation:

$$\Delta u + 1 = 0 \quad \text{in } \Omega,$$

with the boundary conditions:

$$u = 0, \qquad \frac{\partial u}{\partial n} = const. \quad \text{on} \partial\Omega.$$

Based on the idea of moving parallel planes, he proved that if the boundary $\partial\Omega$ is C^2, then the domain Ω is necessarily a ball, and the solution u is radially symmetric and has the specific form $(R^2 - r^2)/2n$, where

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R is the radius of the ball and r denotes distance from its center. In fact, the conclusion still holds when the boundary $\partial\Omega$ is C^1, which is the lowest regularity of the boundary one needed to get Serrin's result (see [42]). There Serrin's result was proved with the aid of Green's theorem and applying the maximum principle to the auxiliary function $|\nabla u|^2 + 2/nu$. Moreover, the paper [7] gave an alternative proof of the Serrin-type overdetermined problem, and their proof did not rely explicitly on the maximum principle. In fact, they applied Newton inequalities for the elementary symmetric functions of the eigenvalues of the matrix D^2u with the help of the Pohožaev identity. Additionally, they generalized this symmetry result to a class of fully nonlinear equations. Recently, H. Shahgholian [37] discovered another proof of Serrin's symmetry problem with a distinct boundary condition described by the distance between two different level surfaces.

This Serrin-type problem on the exterior domain has also been well researched. For example, Reichel [35] proved that for any connected set Ω_1, if there exists a solution $u \in C^2(\Omega)$ for the problem

$$\begin{cases} -\Delta u = f(u), & \Omega = \mathbb{R}^n \setminus \Omega_1, \\ u = a > 0, & \partial\Omega_1, \\ u(x) \to 0, & \text{as } |x| \to \infty, \\ \dfrac{\partial u}{\partial \eta} = c, & \partial\Omega_1, \\ 0 \le u < a, & \Omega, \end{cases}$$

where η is the outer normal vector of Ω_1 and $f(t)$ is a locally Lipschitz function, and nonincreasing for nonnegative and small t. Then the domain is the exterior part of a ball and the solution is radially symmetric. Then Aftalion and Busca [1] removed the decay assumption at infinity and obtained a similar result for the problem with the nonlinear term $f(t)$ satisfying the condition $t^{-\frac{n+2}{n-2}}f(t)$ is nonincreasing. Then the similar problem on annuli-like sets and more complicated multiply-connected sets is considered in [39]. Some related results, we can also refer to [2,43,32–34].

During the last few years, the nonlinear equation involving fractional Laplacian, i.e. $(-\Delta)^\alpha$, has been researched by many mathematicians. Here the nonlocal pseudo-differential operator $(-\Delta)^\alpha$ is defined as

$$(-\Delta)^\alpha u(x) = C_{n,\alpha} PV. \int_{\mathbb{R}^n} \frac{u(x) - u(x)}{|x-y|^{n+2\alpha}} dy,$$

where $0 < \alpha < 1$, $C_{n,\alpha}$ is a constant depending on n and α, and PV denotes the Cauchy Principle Value of the integral (see [30,24]). This nonlocal operator has been used extensively in physics, probability and finance. For the details and some relative results, we refer the readers to [3,6,13,22,10,8]. And the qualitative property, especially the symmetry property for the solution has also been researched extensively, see.e.g. [14,27,29,25,21,26,40]. Particularly, Fall and Jarohs [26] considered the following fractional Laplacian equation:

$$\begin{cases} (-\Delta)^\alpha u = f(u), & \Omega, \\ u = 0, & \mathbb{R}^n \setminus \Omega, \\ (\partial_\eta)_\alpha u = const., & \partial\Omega \end{cases}$$

where $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^n$, $n \ge 1$, is a bounded set with C^2 boundary, η is the outer unit normal vector of $\partial\Omega$ and for any $x \in \partial\Omega$,

$$(\partial_\eta)_\alpha u(x) = \lim_{t\to 0} \frac{u(x - t\eta(x))}{t^\alpha}.$$

The authors obtained the radial symmetry of the domain and the nonnegative nontrivial solution for any locally Lipschitz function f. For the case $n = 2$, $\alpha = 1/2$ and $f = 1$, we can also refer to [23]. Afterwards,