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# Soliton to the fractional Yamabe flow

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ABSTRACT

In this paper, we show that soliton to the fractional Yamabe flow must have constant fractional order curvature.

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## 1. Introduction

Suppose that $X$ is an $(n+1)$-dimensional smooth manifold with smooth boundary $M$, where $n \geq 3$. A function $\rho$ is a defining function on the boundary $M$ in $X$ if

$$\rho > 0 \quad \text{in } X, \qquad \rho = 0 \quad \text{on } M, \qquad d\rho \neq 0 \quad \text{on } M.$$

We say that a Riemannian metric $h^+$ on $X$ is conformally compact if, for some defining function $\rho$, the metric $\overline{h} = \rho^2 h^+$ extends smoothly to $\overline{X}$. This induces a conformal class of metrics $\widehat{g} = \overline{h}|_{TM}$ on $M$ as defining function vary. The manifold $(M, [\widehat{g}])$ equipped with the conformal class $[\widehat{g}]$ is called the conformal infinity of $(X, h^+)$.

A metric $h^+$ is called asymptotically hyperbolic if it is conformally compact and its sectional curvature approaches $-1$ at infinity, which is equivalent to $|d\rho|_{\overline{h}} = 1$ on $M$. If $Ric(h^+) = -nh^+$, then we call $(X, h^+)$ a conformally compact Einstein manifold. In these setting, given a representative $\widehat{g}$ of the conformal infinity, there exists a unique defining function $\rho$ such that in a tubular neighborhood near $M$ such that the metric

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$h^+$ has the normal form

$$h^+ = \frac{d\rho^2 + g_\rho}{\rho^2} \tag{1.1}$$

where $g_\rho$ is a one-parametric family of metric of metrics on $M$ satisfying $g_0 = \widehat{g}$.

The conformal fractional Laplacian $P_\gamma^{\widehat{g}}$ is constructed as the Dirichlet-to-Neumann operator for the scattering problem for $(X, h^+)$. In particular, it follows from [17,30] that given $f \in C^\infty(M)$, for all but a discrete set of values $s \in \mathbb{C}$, the generalized eigenvalue problem

$$-\Delta_{h^+} u - s(n-s)u = 0 \quad \text{in } X \tag{1.2}$$

has a solution of the form

$$u = F\rho^{n-s} + G\rho^s, \quad F, G \in C^\infty(\overline{X}), \qquad F|_{\rho=0} = f. \tag{1.3}$$

The scattering operator on $M$ is defined as

$$S_{\widehat{g}}(s)f = G|_M,$$

and it is a meromorphic family of pseudo-differential operators in the whole complex plane.

The conformal fractional Laplacian on $(M, \widehat{g})$ is defined as

$$P_\gamma^{\widehat{g}} = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)} S_{\widehat{g}}\Big(\frac{n}{2} + \gamma\Big).$$

With this normalization, the principal symbol of the operator $P_\gamma^{\widehat{g}}$ is equal to that of the fractional Laplacian $(-\Delta_{\widehat{g}})^\gamma$. The operator $P_\gamma^{\widehat{g}}$ satisfies the following property: under a conformal change of metric

$$g = u^{\frac{4}{n-2\gamma}}\widehat{g}, \quad u > 0,$$

we have

$$P_\gamma^g(\phi) = u^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{\widehat{g}}(u\phi) \tag{1.4}$$

for all smooth functions $\phi$. As proven in [13,17], when $h^+$ is Poincaré–Einstein, $P_1^{\widehat{g}}$ is the conformal Laplacian, $P_2^{\widehat{g}}$ is the Paneitz operator, and $P_k^{\widehat{g}}$, where $k$ are positive integers, are the GJMS operator discovered in [16]. One can then define the fractional order curvature

$$Q_\gamma^{\widehat{g}} = P_\gamma^{\widehat{g}}(1).$$

Hence, for the case when $\gamma = 1$, the fractional order curvature is the scalar curvature.

As an analogy to the Yamabe problem, one can consider the fractional Yamabe problem: Find a metric $g$ conformal to $\widehat{g}$ such that its fractional order curvature $Q_\gamma^g$ is constant. We refer the readers to [8,14,15,27,31] and references therein for results related to the fractional Yamabe problem. See also [7,24,25] for results related to the fractional Nirenberg problem of prescribing fractional order curvature.

Inspired by the Yamabe flow (see [1,2,9,32,33] for results related to the Yamabe flow, and also [6,5,18,20,21] for results related to the CR Yamabe flow), which is a geometric flow introduced to study the Yamabe problem, we consider the fractional Yamabe flow on $M$. This is defined as the evolution of the metric $g = g(t)$:

$$\frac{\partial g}{\partial t} = -(Q_\gamma^g - \overline{Q_\gamma^g})g, \qquad g|_{t=0} = \widehat{g}, \tag{1.5}$$

where $\overline{Q_\gamma^g}$ is the average of the fractional order curvature $Q_\gamma^g$ given by

$$\overline{Q_\gamma^g} = \frac{\int_M Q_\gamma^g dV_g}{\int_M dV_g}. \tag{1.6}$$

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