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Well-posedness of a fractional porous medium equation on an evolving surface

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ABSTRACT

We investigate the existence, uniqueness, and L^1-contractivity of weak solutions to a porous medium equation with fractional diffusion on an evolving hypersurface. To settle the existence, we reformulate the equation as a local problem on a semi-infinite cylinder, regularise the porous medium nonlinearity and truncate the cylinder. Then we pass to the limit first in the truncation parameter and then in the nonlinearity, and the identification of limits is done using the theory of subdifferentials of convex functionals.

In order to facilitate all of this, we begin by studying (in the setting of closed Riemannian manifolds and Sobolev spaces) the fractional Laplace–Beltrami operator which can be seen as the Dirichlet-to-Neumann map of a harmonic extension problem. A truncated harmonic extension problem will also be examined and convergence results to the solution of the harmonic extension will be given. For a technical reason, we will also consider some related extension problems on evolving hypersurfaces which will provide us with the minimal time regularity required on the harmonic extensions in order to properly formulate the moving domain problem. This functional analytic theory is of course independent of the fractional porous medium equation and will be of use generally in the analysis of fractional elliptic and parabolic problems on manifolds.

1. Introduction

For each $t \in [0, T]$, let $\Gamma(t) \subset \mathbb{R}^{d+1}$ be a smooth and compact d-dimensional hypersurface without boundary evolving with a given velocity field $\mathbf{w}$. In this paper, we are interested in the well-posedness of the fractional porous medium equation

$$\begin{aligned} \dot{u}(t) + (-\Delta_{\Gamma(t)})^{1/2}(u^m(t)) + u(t)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t) &= 0 \quad \text{on } \Gamma(t) \\ u(0) &= u_0 \quad \text{on } \Gamma_0 := \Gamma(0) \end{aligned} \tag{1}$$

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for $m \geq 1$, where u_0 is a given initial data, $u^m := |u|^{m-1}u$ as usual, and $(-\Delta_{\Gamma(t)})^{1/2}$ is the square root of the Laplace–Beltrami operator on $\Gamma(t)$, which is a nonlocal first order elliptic pseudodifferential operator [48,50,61,54].

If the fractional Laplacian in (1) is replaced with the ordinary Laplace–Beltrami operator $-\Delta_{\Gamma(t)}$, (1) would be a porous medium equation on an evolving surface. Porous medium equations on stationary domains have, of course, attracted a considerable and well-developed literature. We refer the reader to the book [56] by Vázquez which is a comprehensive study of the mathematical analysis of the equation (and it also contains many references) and results on the porous medium equation on manifolds can be found in [56, Section 11.5] and [12]. We will also say a few words about the non-fractional moving case in the conclusion of this paper. The investigation of *fractional* porous medium equations was instituted in [27] where the authors examined such a problem on $\mathbb{R}^d$ involving the square root of the Laplacian and gave a complete theory of the equation, and indeed, our work is motivated by the results in that paper. There, the existence was proved by discretisation in time of a localised formulation of the equation and then the application of the Crandall–Liggett theorem [24]. Those results were generalised in [28] to a wider range of fractional powers of the Laplacian $(-\Delta)^s$ with exponent $s \in (0, 1)$ on a stationary domain $\Omega \subseteq \mathbb{R}^d$ using the extension method introduced by Caffarelli and Silvestre in [19]. Existence was proved in [13] (for a more general nonlinearity) in a different way through the theory of semigroups and maximal monotone operators. Our model (1) differs from all of the aforementioned works since it is on a moving space.

Other related works in the literature include variants of nonlocal porous medium equations such as those with variable density [47,46] and different fractional operators [9]. We also mention [5,20,53,43] where elliptic fractional problems are studied in the setting of the Laplacian on a bounded domain with Neumann boundary conditions, and [36] where a degenerate parabolic equation arising in crack dynamics is considered, again in the Neumann setting. One can also find numerical and finite element analysis for elliptic and parabolic problems in [44,45]. As is evident, there has been an extraordinary amount of activity in fractional diffusion problems in the last decade or so. A good survey of recent and current output involving nonlinear fractional diffusion can be found in the articles [57,58].

In terms of the analysis, a common preliminary step when working with half-Laplacians is to rewrite the problem locally using a Dirichlet-to-Neumann map [18,7,51,23]. We will also reformulate (1) using such a map; this step is likewise performed in [27,28] but from here on, the type of approaches taken in [27,28] are problematic in our setting because of the additional complexity engendered by the evolving domain. For example, one could attempt to pull back the problem onto a reference domain (the resulting expression is not too cumbersome if the evolution of $\Gamma(t)$ is prescribed particularly agreeably) and try to employ an appropriate time-dependent version of Crandall–Liggett [25,31,39] to the resulting equation (which will have time-dependent coefficients) but these theorems are difficult to apply even when the evolution of the domains is highly simplified. Therefore, we choose a different way to approach this problem, which we shall outline below, starting from the foundations. To our knowledge, the type of approach developed in this paper has not been used before in the fractional setting, even in the stationary case. The challenges and peculiarities that arise due to the moving domain will be highlighted in due course.

Before we proceed, let us remark that fractional Laplace–Beltrami operators on various classes of manifolds have been studied in [7,51,23] through extension problems in the style of Caffarelli–Silvestre [19], but a convenient work detailing all the relevant properties of the half-Laplacian on closed manifolds in a Sobolev space setting appears lacking, so this paper is useful also in this respect. With this in mind, it is worth emphasising that the first part of this paper, comprising of Sections 2–4, is independent of the second part which consists of Sections 5 and 6, and indeed the reader can read the first part in isolation. The first part can be of use for other fractional diffusion problems on (evolving) manifolds and the second part can be thought of as an application of the first part. See the outline below for more details.