Nonlinear Analysis 137 (2016) 43–76

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Some qualitative properties for geometric flows and its Euler implicit discretization

Luis Alvarez[a], Gregorio Díaz[b], Jesús Ildefonso Díaz[b,*]

[a] *CTIM: Centro de Tec. de la Imagen, Dpto. de Informática y Sistemas, Universidad de Las Palmas de G.C., Campus de Tafira 35017, Spain*

[b] *Instituto de Matemática Interdisciplinar, Depto. de Matemática Aplicada, Universidad Complutense de Madrid, Parque de Ciencias 3, 28040 Madrid, Spain*

ARTICLE INFO

Article history:
Received 7 September 2015
Accepted 19 November 2015
Communicated by S. Carl

Dedicated to an outstanding mathematician near his 70s: Juan Luis Vázquez

MSC:
35K65
35J60
53C44
35K55
35L60
35L81

Keywords:
Geometric flow
Implicit discretization
Nonlinear parabolic and elliptic equations

ABSTRACT

We study the geometric flow parabolic equation and its implicit discretization which yield a family of nonlinear elliptic problems. We show that there are important differences in the study of those equations which concerns the propagation of level sets of data. Our study is based on the previous study of radially symmetric solutions of the corresponding equation. Curiously, in radial coordinates both equations reduce to suitable singular Hamilton–Jacobi first order equations. After considering the case of monotone data we point out a new peculiar behavior for non-monotone data with a profile of *Batman* type ($g = \min\{g_1, g_2\}, g_1(r)$ increasing, $g_2(r)$ decreasing and $g_1(r_d) = g_2(r_d)$ for some $r_d > 0$). In the parabolic regime, and when the velocity of the convexity part of the level sets is greater than the velocity of the concavity part, we show that the level set $\{u = g(r_d)\}$ develops a non-empty interior set for any $t > 0$. Nothing similar occurs in the stationary regime. We also present some numerical experiences.

1. Introduction

This paper deals with several qualitative properties of geometric flows presented here, for simplicity in dimension 2, while all of our results admit generalizations to higher dimensions. A geometric flow can be defined as an operator $\mathrm{T}_t(f)$ which provides, for an original function f, a family of functions $\mathrm{T}_t(f)$ with $t \geq 0$. The geometric character of the flow is due to the assumption that the level set evolution of function f depends just on the geometry of the boundary of the level sets. We can formalize this geometric character

* Corresponding author.
E-mail addresses: lalvarez@ulpgc.es (L. Alvarez), gdiaz@ucm.es (G. Díaz), jidiaz@ucm.es (J.I. Díaz).

http://dx.doi.org/10.1016/j.na.2015.11.023
0362-546X/

using the named *morphological invariance assumption* which can be expressed as:

$$\mathrm{T}_t(f) \circ h = \mathrm{T}_t(f \circ h) \tag{1}$$

for any increasing function $h(\cdot)$. As it was proved in [6] (see also [35]) and [1], under some minimal architectural assumptions, all the geometric flows are generated by the partial differential equation:

$$\frac{\partial u}{\partial t} = \beta(\mathrm{curv}(u))|\mathrm{D}u|, \tag{2}$$

where $\mathrm{curv}(u)(x)$ is the curvature of the level line containing the point x, that is:

$$\mathrm{curv}(u) = \mathrm{div}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|}\right), \tag{3}$$

and $\beta(\cdot)$ is a nondecreasing function given by:

$$\beta(s) = \begin{cases} b_+ s^q, & \text{if } s \geq 0, \\ -b_-\,(-s)^q\,, & \text{if } s < 0, \end{cases} \tag{4}$$

with $q, b_+, b_- \geq 0$ such that $b_+ + b_- > 0$. Therefore the geometric flow depends on three parameters, q, b_-, b_+. Among the different choices of these parameters we point out the case $q = 1$, $b_- = b_+ = 1$, which corresponds to the *mean curvature operator* and $q = 1/3$, $b_- = b_+ = 1$, which corresponds to the *affine invariant equation* studied in [6] (in some works, as for instance [32, page 35], this last case is considered by assuming $b_- = 0$).

If $u(t,x)$ is the solution of Eq. (2), for the initial datum f, then $u(t,x) = \mathrm{T}_t(f)(x)$ and thus T_t represents the semigroup associated to the parabolic problem. For a level $l \in \mathbb{R}$, the associated level set of function f is defined as $L_f(l) = \overline{\{x : \ f(x) < l\}}$. The geometric character of the flow means that the evolution of a level set $L_f(l)$, given by $L_{\mathrm{T}_t(f)}(l)$, depends just on the geometry of its boundary $\partial L_f(l)$. Due to its morphological invariance, geometric flows are commonly used in *Computer Vision* applications (see, *e.g.*, [37,1] for an application to shape representation and [36,2] for the use of affine invariant distances to a curve).

Our main interest in this paper will be centered in to get some new qualitative properties of solutions of the parabolic problem

$$\mathrm{PP}(\mathbb{R}^2) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \beta\left(\mathrm{div}\left(\dfrac{\mathrm{D}u}{|\mathrm{D}u|}\right)\right)|\mathrm{D}u| = 0 & \text{in } (0,T)\times\mathbb{R}^2, \\ u(0,\cdot) = u_0(\cdot) & \text{on } \mathbb{R}^2. \end{cases}$$

Notice that the above mentioned initial datum (the original *image function* f) is now denoted as u_0. We shall also pay attention to the case of an bounded regular domain $\Omega \subset \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions. A typical example is the case of homogeneous boundary conditions

$$\mathrm{PP}(\Omega) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \beta\left(\mathrm{div}\left(\dfrac{\mathrm{D}u}{|\mathrm{D}u|}\right)\right)|\mathrm{D}u| = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0(\cdot) & \text{on } \Omega, \end{cases}$$

but weshall also consider the case of non-homogeneous boundary data (see Corollary 1).

In some sense, this paper can be understood as continuation of several previous papers by the authors in which the qualitative properties of solutions of different parabolic problems are analyzed jointly with the suitable understanding of the qualitative properties of solutions of the stationary problems resulting from their the implicit Euler discretization. The main connection among the family of the diverse classes of considered problems comes from the fact that the involved *elliptic* operators generate a semigroup of contractions in some Banach space and thus, thanks to the abstract semigroup theory, the convergence of the *discretized solutions* is ensured in the corresponding Banach space. This point of view was applied