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Nonuniqueness of solutions for a class of forward–backward parabolic equations

Michiel Bertsch[a,b], Flavia Smarrazzo[c], Alberto Tesei[d,b,*]

[a] *Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy*

[b] *Istituto per le Applicazioni del Calcolo "M. Picone", CNR, Roma, Italy*

[c] *Università Campus Bio-Medico di Roma, Via Alvaro del Portillo 21, 00128 Roma, Italy*

[d] *Dipartimento di Matematica "G. Castelnuovo", Università "Sapienza" di Roma, P.le A. Moro 5, I-00185 Roma, Italy*

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A B S T R A C T

We study the initial–boundary value problem

$$\begin{cases} u_t = [\varphi(u)]_{xx} + \varepsilon[\psi(u)]_{txx} & \text{in } \Omega \times (0,\infty) \\ \varphi(u) + \varepsilon[\psi(u)]_t = 0 & \text{in } \partial\Omega \times (0,\infty) \\ u = u_0 \geq 0 & \text{in } \Omega \times \{0\} \end{cases}$$

with measure-valued initial data. Here Ω is a bounded open interval, $\varphi(0) = \varphi(\infty) = 0$, φ is increasing in $(0,\alpha)$ and decreasing in (α,∞), and the regularising term ψ is increasing but bounded. It is natural to study measure-valued solutions since singularities may appear spontaneously in finite time. Nonnegative Radon measure-valued solutions are known to exist and their construction is based on an approximation procedure. Until now nothing was known about their uniqueness.

In this note we construct some nontrivial examples of solutions which do not satisfy all properties of the constructed solutions, whence uniqueness fails. In addition, we classify the steady state solutions.

1. Introduction

In this note we consider the partial differential equation

$$u_t = [\varphi(u)]_{xx} + \varepsilon[\psi(u)]_{txx} \quad \text{in } Q := \Omega \times (0,\infty), \tag{1.1}$$

* Corresponding author at: Dipartimento di Matematica "G. Castelnuovo", Università "Sapienza" di Roma, P.le A. Moro 5, I-00185 Roma, Italy.

E-mail addresses: bertsch.michiel@gmail.com (M. Bertsch), flavia.smarrazzo@gmail.com (F. Smarrazzo), albertotesei@gmail.com (A. Tesei).

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with initial–boundary conditions

$$\varphi(u) + \varepsilon[\psi(u)]_t = 0 \quad \text{in } \partial\Omega \times (0,\infty), \ u = u_0 \geq 0 \text{ in } \Omega \times \{0\}. \tag{1.2}$$

Here ε is a positive constant, $\Omega \equiv (a,b) \subset \mathbb{R}$ is a bounded interval and u_0 is a nonnegative Radon measure on Ω. The function $\varphi : [0,\infty) \to [0,\infty)$ is increasing in $(0,\alpha)$ and decreasing in (α,∞) for some $\alpha > 0$, and $\varphi(0) = \varphi(\infty) = 0$. The function ψ is increasing and bounded, $\psi(0) = 0$, $\psi(\infty) = \gamma > 0$ and $\psi'(u) \to 0$ as $u \to \infty$, whence (1.1) is a *strongly degenerate pseudo-parabolic equation*. Possible choices of φ and ψ are

$$\varphi(u) = \frac{u}{u^2+\alpha^2}, \qquad \psi(u) = \gamma\left[1-(1+u)^{-\sigma}\right] \quad (\alpha,\sigma > 0). \tag{1.3}$$

For the precise hypotheses on φ and ψ we refer to (H_1)–(H_2).

Eq. (1.1) is a regularisation of the forward–backward parabolic equation $u_t = [\varphi(u)]_{xx}$, which is related to the Perona–Malik equation [10]

$$z_t = [\varphi(z_x)]_x \tag{1.4}$$

through the identification $u = z_x$. The strongly degenerate pseudo-parabolic regularisation of (1.4) arises naturally in a mathematical model for the formation of layers of constant temperature or salinity in the ocean (with $z_x \geq 0$, see [1,2]). It was already observed in [2] that solutions may become singular in finite time. In terms of z this means that solutions may become discontinuous in x, and in terms of $u = z_x$ that Dirac masses may appear in finite time, so solutions are intrinsically measure-valued. In a forthcoming paper [5] we shall study the process of spontaneous singularity formation in detail and show that more general nonnegative Radon measures, even singular continuous ones, may appear in finite time. The formation of singularities is obviously related to the specific shape of φ, but also to the boundedness of ψ. Apparently the degeneracy of the regularising term is so strong that it does not prevent singularity formation. It is known that singularities cannot appear spontaneously if ψ has algebraic [4] or logarithmic [3] growth at ∞.

Since φ and ψ are nonlinear functions, it is not obvious how to define a measure-valued solution. Below (see Definition 3.1) we shall specify what we mean by a solution of problem (1.1)–(1.2). According to the definition, the map $t \mapsto u(\cdot,t)$ describes, roughly speaking, a continuous orbit in the space of nonnegative Radon measures and satisfies the equation $u_t = v_{xx}$ in the sense of distributions, where the *chemical potential* $v = v(x,t)$ is a bounded function which is continuous with respect to x. To give a mathematical meaning to the formal relation $v = \varphi(u) + \varepsilon[\psi(u)]_t$ if u is measure-valued, we denote by $u_{ac}(\cdot,t)$ and $u_s(\cdot,t)$ the absolutely continuous and singular parts of $u(\cdot,t)$, and by $\mathcal{B}(t) \subset \overline{\Omega} \setminus \operatorname{supp} u_s(\cdot,t)$ the set where the density $u_r(\cdot,t)$ of $u_{ac}(\cdot,t)$ is locally bounded. It is reasonable to require that, at least for almost every $t > 0$,

$$v(\cdot,t) = \varphi(u_r(\cdot,t)) + \varepsilon[\psi(u_r(\cdot,t))]_t \quad \text{almost everywhere in } \mathcal{B}(t). \tag{1.5}$$

On the other hand, if one has in mind that $\varphi(\infty) = 0$ and $\psi(\infty) = \gamma$ and heuristically thinks the equality $v(\cdot,t) = \varphi(u(\cdot,t)) + \varepsilon[\psi(u(\cdot,t))]_t$ to be valid in all of Ω, it is not surprising either to require that

$$v(\cdot,t) = 0 \quad \text{almost everywhere in } \overline{\Omega} \setminus \mathcal{B}(t). \tag{1.6}$$

Relations (1.5) and (1.6) are satisfied by any solution of problem (1.1)–(1.2) in the sense of Definition 3.1 ([8]; see (3.9)–(3.10)). In particular, they are satisfied by *constructed solutions* of the problem (whose existence is proven by [7, Theorem 3.7]), which are obtained by a natural approximation procedure applied to both ψ and u_0.

In this paper we show that the solution concept given by Definition 3.1 does not guarantee uniqueness of solutions. To prove the nonuniqueness statement we construct examples of solutions which do not satisfy *all* properties of the constructed solutions. In addition, as we shall see in Section 5 (see also the concluding remarks in Section 5.3), these examples are instructive since they give more insight in how singularities can be created. They also give a first hint for a uniqueness criterion, a problem which is completely open.