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Compactness of minimizing sequences

Lucio Boccardo [a,*], Thierry Gallouet [b]

[a] *Dipartimento di Matematica, Sapienza Università di Roma, Piazza A. Moro 2, Roma, Italy*
[b] *Institut de Mathematiques de Marseille, 39 rue Joliot Curie, 13453 Marseille cedex 13, France*

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ABSTRACT

We consider a minimization problem of a functional in the space $W_0^{1,p}(\Omega)$, where $1 < p < +\infty$ and Ω is a bounded open set of $\mathbb{R}^N$. We prove the compactness, in the space $W_0^{1,p}(\Omega)$, under convenient hypotheses, of a minimizing sequence. The main difficulty is to prove the convergence in measure of the gradient of the minimizing sequence. Furthermore, considering a sequence of minimization problems in the space $W_0^{1,p}(\Omega)$, we prove some convergence results of the sequence of minimizers to the minimizer of the limit problem.

1. Introduction and main results

We deal with integral problems where the functional is defined as

$$J(v) = \int_\Omega j(x, v, \nabla v) - \int_\Omega f v, \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, and $j : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, that is, measurable with respect to x in Ω for every $(s, \xi)\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω.

We assume that there exist $g \in L^1(\Omega)$ and real positive constants α, β such that for almost every x in Ω, for every s in $\mathbb{R}$, for every ξ and η in $\mathbb{R}^N$ we have

$$\alpha|\xi|^p \leq j(x, s, \xi), \tag{2}$$

$$j(x, s, \xi) \leq \beta(|\xi|^p + |s|^p) + g(x), \tag{3}$$

$$f(x) \in L^m(\Omega), \quad m \geq (p^\star)', \tag{4}$$

* Corresponding author.
E-mail addresses: boccardo@mat.uniroma1.it (L. Boccardo), thierry.gallouet@univ-amu.fr (T. Gallouet).

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where $1 < p$, $(p^\star)'$ is the Sobolev conjugate of p, if $1 < p < N$, it is any number greater than 1 if $p = N$, and $m = 1$ if $p > N$.

Thus $J(v)$ is well defined in $W_0^{1,p}(\Omega)$.

Theorem 1. *We assume (2), (3), (4) and*

$$j\big(x, s, \xi\big) \text{ is strictly convex with respect to } \xi, \tag{5}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Then the minimizing sequences of J, defined in (1), are compact in $W_0^{1,p}(\Omega)$. Furthermore, if u is a limit of a minimizing sequence, then it is a minimizer of J.

The situation, described in Theorem 1 is known in the Calculus of Variations, in some simple cases, where it is easy to prove that a weakly convergent minimizing sequence is also strongly convergent (see Remark 4). Our approach uses deeply Real Analysis techniques and it is slightly close to a method used in [4].

Moreover, we point out some relationships with the results of the papers [5,8,7]. In [5], is proved that, under some assumptions on the strictly convex function $j : \mathbb{R}^M \to \mathbb{R}$, if $(u_n)_{n\in\mathbb{N}}$ and u are functions in $L^1(\Omega, \mathbb{R}^M)$, the sequence (u_n) converges weakly in $\mathcal{D}'$ (convergence assumption weaker than the assumption of the previous papers) and $\limsup \int_\Omega j(u_n) \le \int_\Omega j(u)$, then (u_n) converges strongly in $L^1(\Omega, \mathbb{R}^M)$.

Theorem 1 is also true if Hypothesis (4) is replaced by $f \in W^{-1,p'}(\Omega)$ with $p' = p/(p-1)$ and, in (1), $\int_\Omega fv$ is replaced by the duality product between f and v. We prove Theorem 1 in Section 2.

An adaptation of the proof of Theorem 1 gives the following result on the convergence of the sequence of minimizers associated to a sequence of data $(f_n)_{n\in\mathbb{N}}$. We denote by $\langle\cdot,\cdot\rangle$ the duality product between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Theorem 2. *We assume (2), (3) and (5). We assume furthermore that j does not depend on its second argument. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $W^{-1,p'}(\Omega)$ and f such that*

$$f_n \text{ converges to } f \text{ in } W^{-1,p'}(\Omega), \quad \text{as } n \to \infty. \tag{6}$$

Let u be the minimizer (in $W_0^{1,p}(\Omega)$) of $\int_\Omega j(x, \nabla v) - \langle f, v\rangle$ and, for all n, let u_n be the minimizer (in $W_0^{1,p}(\Omega)$) of $\int_\Omega j(x, \nabla u_n) - \langle f_n, v\rangle$.

Then the sequence $\{u_n\}$ converges to u in $W_0^{1,p}(\Omega)$.

In Theorem 2, the existence of u (and of u_n for all n) is an easy consequence of (2), (3), (5). In order to prove the uniqueness of u (and of u_n for all n) we also use the fact that j does not depend on its second argument. Indeed, let $v, w \in W_0^{1,p}(\Omega)$ such that $v \ne w$. Let $A = \{\nabla v \ne \nabla w\}$. One has, thanks to (5),

$$j\left(\cdot, \frac{1}{2}\nabla v + \frac{1}{2}\nabla w\right) < \frac{1}{2}j(\cdot, \nabla v) + \frac{1}{2}j(\cdot, \nabla w) \quad \text{a.e. on } A.$$

Then, since the measure of A is positive, this gives $J(\frac{1}{2}\nabla v + \frac{1}{2}\nabla w) < \frac{1}{2}J(v) + \frac{1}{2}J(w)$ and proves the uniqueness of the minimizers in Theorem 2.

Finally, the proof of the convergence of u_n to u in $W_0^{1,p}(\Omega)$ is given in Section 3.

A natural question consists to replace in Theorem 2 the hypothesis (6) by the hypothesis

$$f_n \text{ converges to } f \text{ weakly in } W^{-1,p'}(\Omega), \quad \text{as } n \to \infty. \tag{7}$$

If $p = 2$, the conclusion of Theorem 2 becomes that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$. This is quite easy to prove, thanks to fact that the Euler–Lagrange equation of this minimization problem is linear. If $p \ne 2$, this result is not true. A counter example is given in Section 4. However, we have a convergence result of u_n to u, with an additional hypothesis on the sequence $(f_n)_{n\in\mathbb{N}}$. This is given in Theorem 3, whose proof is also in Section 3.