Nonlinear Analysis 134 (2016) 70–104

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

An axiomatic approach to gradients with applications to Dirichlet and obstacle problems beyond function spaces

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A R T I C L E I N F O

Article history:
Received 24 August 2015
Accepted 11 December 2015
Communicated by S. Carl

MSC:
primary 49J27
secondary 31E05
35J50
35J66
49Q20
46L51
46L52

Keywords:
Dirichlet problem
First eigenvalue
Generalized Sobolev space
Gradient relation
Lattice
Metric space
Noncommutative function
Obstacle problem
Operator-valued function
Partial order
Poincaré set
Rayleigh quotient
Rellich–Kondrachov cone
Trace class ideal
Variational problem

A B S T R A C T

We develop a framework for studying variational problems in Banach spaces with respect to gradient relations, which encompasses many of the notions of generalized gradients that appear in the literature. We stress the fact that our approach is not dependent on function spaces and therefore applies equally well to functions on metric spaces as to operator algebras. In particular, we consider analogues of Dirichlet and obstacle problems, as well as first eigenvalue problems, and formulate conditions for the existence of solutions and their uniqueness. Moreover, we investigate to what extent a lattice structure may be introduced on (ordered) Banach spaces via a norm-minimizing variational problem. A multitude of examples is provided to illustrate the versatility of our approach.

1. Introduction

In the classical theory of partial differential equations, one explores the existence of solutions (and their regularity) by extending spaces of differentiable functions to include functions with only a weak

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http://dx.doi.org/10.1016/j.na.2015.12.010

notion of derivative. Introducing L^p-spaces and Sobolev spaces has the advantage that one may exploit the completeness of these spaces in order to find weak solutions of differential equations. In doing so, one is forced to work with equivalence classes of functions, rather than single functions, and the classical value of a function at a point is, for some purposes, simply not relevant anymore. Consequently, one tends to use Banach space techniques to reach the desired results. In particular, when extending the theory to functions on more general spaces, it becomes apparent that abstract methods are useful as classical techniques may not be applicable.

Consider the Dirichlet problem for harmonic functions, i.e. to find a harmonic function with given boundary values in a bounded domain Ω in $\mathbf{R}^n$. This problem can equivalently be reformulated as finding the minimizer of the energy integral

$$\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega |\nabla u|^2\,dx, \tag{1.1}$$

over all sufficiently smooth functions with given boundary values. In this note, we aim to give an axiomatic approach to such problems starting from a quite general notion of gradient, assuming only a weak form of linearity. Many particular examples of gradients, such as weak gradients, upper gradients in metric spaces, Hajłasz gradients and algebraic derivations, fall into this class. We shall also consider gradients with no relation to derivatives (cf. Section 8), as well as examples which come from higher-order differential operators, such as the Laplacian and biLaplacian (cf. Section 8.3). It deserves to be pointed out that the framework we develop depends neither on function spaces nor on the commutativity of multiplication and, therefore, applies equally well to noncommutative settings, such as operator algebras.

We start by introducing an abstract notion of gradient relation and define a Sobolev space based on it. We show that, under minimal assumptions, this generalized Sobolev space is always a Banach space and that functions therein possess a unique minimal gradient. In Theorem 3.2 we formulate sufficient conditions for the existence of solutions to the Dirichlet problem with respect to this minimal gradient in analogy with (1.1). Furthermore, in Proposition 3.4 we give a condition for the solution to be unique.

In addition to the Dirichlet problem, we also consider the obstacle problem as well as the first eigenvalue problem (strictly speaking the existence of minimizers for the Rayleigh quotient, cf. Theorem 5.3). To solve the obstacle problem we reformulate it as a Dirichlet problem, and we can thus use the Dirichlet problem theory to solve the obstacle problem. Already here one can see the power of our abstract approach, as one can rarely consider obstacle problems as special cases of Dirichlet problems in more traditional situations (cf. Remark 3.5). A prominent role in the minimization problems above is played by *Poincaré sets*, i.e. subsets $\mathcal{K}$ of the abstract Sobolev space which support a generalized Poincaré inequality:

$$\|u\| \le C\,\|\nabla u\|\,, \quad u \in \mathcal{K}.$$

Such sets provide natural domains when considering variational problems in the context of gradient relations.

Finally, inspired by the fact that the pointwise maximum of two functions minimizes the L^p-norm among all functions which majorize both functions, we investigate the possibility of defining the maximum (as well as the minimum) of two elements in a Banach space via a minimization problem (cf. Propositions 6.4 and 6.12). Furthermore, we formulate necessary and sufficient conditions for the existence of least upper (resp. greatest lower) bounds (cf. Theorem 6.15).

An axiomatic approach to gradient structures has also been given by Gol'dshtein–Troyanov [5], in the setting of metric spaces. In contrast to our approach, their theory is based on function spaces, more precisely on L^p spaces, and is thus considerably more restrictive than ours. Even for L^p, $p > 1$, our axioms allow for a much larger class of generalized gradient structures. (For L^1 their assumptions can be fulfilled while ours cannot.) The axiomatic theory from [5] has been further developed in Gol'dshtein–Troyanov [6] and Timoshin [27,28], but neither the Dirichlet problem nor the obstacle problem has been considered.