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Existence and uniqueness of monotone nodal solutions of a semilinear Neumann problem

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ABSTRACT

In this paper, we study monotone radially symmetric solutions of semilinear equations with Allen–Cahn type nonlinearities by the bifurcation method. Under suitable conditions imposed on the nonlinearities, we show that the structure of the monotone nodal solutions consists of a continuous U-shaped curve bifurcating from the trivial solution at the third eigenvalue of the Laplacian. The upper branch consists of a decreasing solution and the lower branch consists of an increasing solution. In particular, we show the following equation

$$\Delta u + \lambda(u - u \mid u \mid^{p-1}) = 0 \quad \text{in } B, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B$$

has exactly two monotone radial nodal solutions, one is decreasing and the other is increasing. Here B is the unit ball in $\mathbb{R}^n$, $p > 1$ and $\lambda > 0$.

1. Introduction

This paper is concerned with the uniqueness of the nodal radial symmetry monotone solution of the Allen–Cahn equation

$$\Delta u + \lambda f(u) = 0 \tag{1.1}$$

on the unit ball B in $\mathbb{R}^n$, $n \geq 2$, and $\lambda > 0$ is a parameter.

Problem (1.1) serves as a model in many different areas of applied mathematics and has been extensively investigated in the last three decades. For example, it appears in astrophysics as the nonlinear scalar field equation; in chemistry and population dynamics governing the stationary states; in fluid mechanics describing the blowup set of some porous-medium equations, and also in some models in plasma physics.

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In the case of the Dirichlet boundary condition, the problem is

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth domain in $\mathbb{R}^n$, and when Ω is unbounded, the boundary condition will be understood as $u(x) \to 0$ when $|x| \to \infty$. The existence of a positive solution of (1.2) has been treated by many authors; see [11,7]. When $\Omega = B$ or $\Omega = \mathbb{R}^n$, it is well known [14,15,20] that all positive solutions must be radially symmetric (up to translation), provided certain conditions on f hold. A monotone decreasing radially symmetric bounded entire solution is called a ground state solution, and the existence of a ground state solution was proved by [32,1,12,13]. Many authors focused on the uniqueness of positive solutions in a finite ball, an annulus or the entire space $\mathbb{R}^n$; see [8,38,39,28,18,19,27,9,37,16,13].

For the Neumann boundary condition, we use the convention to set $\varepsilon^2 = 1/\lambda$. Hence the problem becomes

$$\begin{cases} \varepsilon^2 \Delta u + f(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ and ν is the unit outward normal vector on $\partial\Omega$. Problem (1.3) usually regards as a singularly perturbed nonlinear equation, and it has some meaningful phenomenon such as concentrating solutions (solutions concentrate at one or more points in Ω as $\varepsilon \to 0$) for small ε. It has attracted a lot of attention in the past thirty years since Lin and Ni initiated the study of qualitative properties of Neumann problems. The least energy solutions [35] and spike layer solutions were important advances in the field. The pioneering works [34,22,23,35,36] on a single-peak spike-layer solution studied the shape of the least energy solution. Lin studied the radial symmetry and axial symmetry of the shape of least energy solution when the domain is a ball; see [21,25,6]. In [40,41,24,42], the focus is on the multi-spike layer solutions. Most of them only consider singular perturbation problems, namely the small diffusion coefficient case. Up until now, the behavior of least energy solutions is not well understood when the diffusion coefficient is not small enough. In general, the least energy solutions should lie on the solution curve that bifurcates at the second Neumann eigenvalue. In order to analyze all least energy solutions of (1.3), it is natural to consider the problem from the bifurcation point of view. Along this line, Miyamoto studied the structure of the radially symmetric decreasing solutions and axis symmetric solutions of (1.3) for $\Omega = B$, which bifurcate from the trivial solution at the second eigenvalue (see [29–31]). For existence of radial nondecreasing solutions for the Neumann problems, we refer readers to [4,2,3]. To study the positive solutions from the bifurcation point of view, we refer readers to [17].

The aim of our paper is to give more complete information on the structure of monotone radial nodal solutions for Allen–Cahn type nonlinearities. We use the bifurcation method to show that the structure of the monotone nodal solutions consists of a continuous U-shaped curve which bifurcates from the trivial solution at the second Neumann radial eigenvalue μ_2 of Laplacian. The upper branch consists of decreasing solutions and the lower branch consists of increasing solutions, In particular, we have the following:

Theorem 1.1. *Let $\varepsilon = 1$ and Ω be the unit ball. Assume f satisfies* (f1)–(f2) *below.* (1.3) *does not have any monotone radial solution for $f'(0) < \mu_2$;* (1.3) *admits exactly two monotone radial nodal solutions if $f'(0) > \mu_2$; Furthermore, one is decreasing and the other is increasing.*

Consider the following parameterized equation

$$\Delta u + \lambda f(u) = 0 \quad \text{in } B, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B, \tag{1.4}$$

or

$$u'' + \frac{n-1}{r} u' + \lambda f(u) = 0 \quad \text{in } (0,1), \qquad u'(0) = u'(1) = 0 \tag{1.5}$$