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Semilinear elliptic equations and nonlinearities with zeros

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ABSTRACT

In this paper we consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is a nonnegative, locally Lipschitz continuous function, Ω is a smooth bounded domain and $\lambda > 0$ is a parameter. Under the assumption that f has an isolated positive zero α such that

$$\frac{f(t)}{(t-\alpha)^{\frac{N+2}{N-2}}} \text{ is decreasing in } (\alpha, \alpha+\delta),$$

for some small $\delta > 0$, we show that for large enough λ there exist at least two positive solutions $u_\lambda < v_\lambda$, verifying $\|u_\lambda\|_\infty < \alpha < \|v_\lambda\|_\infty$ and $u_\lambda, v_\lambda \to \alpha$ uniformly on compact subsets of Ω as $\lambda \to +\infty$. The existence of these solutions holds independently of the behavior of f near zero or infinity.

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1. Introduction and results

The purpose of this paper is the study of the semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a nonnegative, locally Lipschitz function defined in $[0, +\infty)$, Ω is a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ and $\lambda > 0$ will be regarded as a parameter. Our main objective is to analyze the existence and multiplicity of positive classical solutions of (1.1) when λ is large.

When f is positive, it turns out that the behaviors at zero and infinity are important in order to ensure the existence of solutions in some domains. The typical example is $f(t) = t^p$ with $p \geq 1$, where it is well known that the necessary and sufficient condition for existence is $1 < p < \frac{N+2}{N-2}$, at least for star-shaped domains. The situation, however, is slightly different when the nonlinearity has a positive zero; actually, the presence of a zero makes it possible to construct positive solutions without assuming prescribed behaviors for f near zero or infinity. We refer to the survey [17] for an early study of this problem, and to [2,13,14] for its extensions to some more general operators. These previous works were slightly improved recently in [9] where some better conditions on f near its zeros were found to get the existence of at least two solutions for each isolated zero of f when λ is large.

Our objective in the present paper is to generalize the condition found in [9] to deal with some more general nonlinearities. In this regard, it is worthy of mention that, for the canonical class of examples given by $f(t) = h(t)|t-1|^q$, where h is locally Lipschitz and $h(1) > 0$, the condition previously found in the literature (and specifically in [9]) for the existence of two positive solutions when λ is large enough and the maxima are close to 1, is $q \leq \frac{N}{N-2}$. This restriction comes from the fact that a Liouville theorem for supersolutions in $\mathbb{R}^N$ is used at some moment for the construction of the second solution. In the present work we will find a significantly better condition by using a different Liouville theorem. More precisely, to apply it, we will require that f verifies the following weaker hypothesis:

There exists $\delta > 0$ such that the function

$$\frac{f(t)}{(t-\alpha)^{\frac{N+2}{N-2}}}, \tag{H}$$

is decreasing for $t \in (\alpha, \alpha + \delta)$.

For the class of examples alluded to above, this condition holds whenever $1 \leq q < \frac{N+2}{N-2}$ or $q = \frac{N+2}{N-2}$ and h is decreasing near 1. Thus $\frac{N+2}{N-2}$ is the expected sharp exponent.

We would like to stress once again that only the behavior of f near its zero α given by hypothesis (H) will be important to have positive solutions for large λ. Therefore the growth of f at infinity can be arbitrary.

Let us next state our main result.

Theorem 1. *Assume $N \geq 3$, and let f be a nonnegative locally Lipschitz function with an isolated zero $\alpha > 0$, such that hypothesis (H) holds. Then there exists $\lambda_0 > 0$ such that, for $\lambda > \lambda_0$, problem (1.1) admits at least two positive solutions u_λ and v_λ with $u_\lambda < v_\lambda$ in Ω, and $\|u_\lambda\|_\infty < \alpha < \|v_\lambda\|_\infty$. Moreover,*

$$\lim_{\lambda \to +\infty} u_\lambda(x) = \lim_{\lambda \to +\infty} v_\lambda(x) = \alpha,$$

uniformly on compact subsets of Ω.

In particular, for the class of examples given by $f(t) = h(t)|t-1|^q$, with h not vanishing at one, we have: