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Positive solutions with a time-independent boundary singularity of semilinear heat equations in bounded Lipschitz domains

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ABSTRACT

We study time-global positive solutions of semilinear heat equations of the form $u_t - \Delta u = f(x,u)$ in a bounded Lipschitz domain Ω in $\mathbb{R}^n$. In particular, we show the existence of a positive solution with a time-independent singularity at a boundary point ξ of Ω which converges to a positive solution, with the behavior like the Martin kernel at ξ, of the corresponding elliptic equation at time infinity. A nonlinear term f is conditioned in terms of a certain Lipschitz continuity with respect to the second variable and a generalized Kato class associated with the Martin kernel at ξ, and admits not only usual one $V(x)u^p(\log(1+u))^q$, but also one with variable exponents.

1. Introduction

During the last few decades, the existence and the asymptotic behavior of time-global positive solutions, with a time-independent singularity or a time-dependent singularity, of the heat equation with a nonlinear reaction term in the whole space $\mathbb{R}^n$ or in a bounded domain in $\mathbb{R}^n$ have been studied extensively. Now, let Ω be a domain in $\mathbb{R}^n (n \geq 3)$ containing the origin 0, and consider the initial–boundary value problem

$$\begin{cases} u_t - \Delta u = V(x)u^p & \text{in } (\Omega \setminus \{0\}) \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for all } x \in \Omega. \end{cases} \tag{1.1}$$

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Here, $u = u(x,t)$, Δ is the Laplacian with respect to $x \in \mathbb{R}^n$, $u_t = \partial u/\partial t$, V is a nonnegative Borel measurable function on Ω, $p > 1$, u_0 is a nonnegative continuous function on Ω, and the equation $u_t - \Delta u = V(x)u^p$ is understood in the sense of distributions. In [15], Sato proved that, in the case where $\Omega = \mathbb{R}^n$, $V(x) \equiv 1$ and

$$\frac{n}{n-2} < p < \begin{cases} \dfrac{n+2\sqrt{n-1}}{n-4+2\sqrt{n-1}} & \text{if } n \leq 10, \\ \dfrac{n+2}{n-1} & \text{if } n > 10, \end{cases}$$

the problem (1.1) has a time-global positive solution u with a time-independent singularity at the origin such that for each $t > 0$, $\|x\|^{2/(p-1)}u(x,t)$ converges to some constant L depending only on p and n as $x \to 0$, and for each $x \in \mathbb{R}^n \setminus \{0\}$, $u(x,t)$ converges to the singular steady-state $L\|x\|^{-2/(p-1)}$ as $t \to \infty$, whenever the initial value $u_0(x)$ is not greater than $C\|x\|^{-2/(p-1)}$ for some constant $C > 0$. Also, Sato and Yanagida [16,17] investigated, for p and V being the same as above, the existence of time-local and time-global positive solutions with a prescribed time-dependent singularity in $\mathbb{R}^n$ and some properties including a comparison principle, when u_0 behaves like the above singular steady-state. In contrast, when $V(x)$ vanishes continuously at the origin or $1 < p < n/(n-2)$, one can get a singular solution with the different behavior from the above singular steady-state. Before their works, Zhang and Zhao [18] proved the existence of a time-global positive solution, with a time-independent singularity at the origin, of the problem (1.1) in a bounded Lipschitz domain Ω which converges to a singular solution, with the behavior like the fundamental solution of Laplace's equation, of the corresponding elliptic problem at time infinity. It is noteworthy that their arguments from the point of view of potential theory enable us to treat a general potential V and a nonsmooth domain Ω. Later, Riahi [14] generalized a potential class and refined their arguments to give a simpler proof. Also, in the recent papers due to Kan and Takahashi [10,11], the existence and the behavior of positive solutions of $u_t - \Delta u = u^p$ having a prescribed time-dependent singularity in the case $1 < p < n/(n-2)$ are studied. We refer to Karch and Zheng [12] for the Navier–Stokes system.

As far as I know, there is no result concerning the existence of time-global positive solutions with singularities on the boundary. This problem requires more delicate estimates in our analysis, because a singularity is influenced by the shape of a domain, and is more difficult than the problem of an interior singularity. The purpose of this paper is to show the existence of a positive solution $u = u(x,t)$ with a time-independent singularity at $\xi \in \partial\Omega$, the boundary of Ω, of the initial–boundary value problem for the following semilinear heat equation:

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } (\partial\Omega \setminus \{\xi\}) \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for all } x \in \Omega, \end{cases} \tag{1.2}$$

where f is a nonnegative Borel measurable function on $\Omega \times [0,\infty)$ satisfying weak conditions stated below, and the equation $u_t - \Delta u = f(x,u)$ is understood in the sense of distributions. Let $M_\Omega(\cdot,\xi)$ be the Martin (Poisson) kernel on Ω with pole at ξ and let $\mathcal{K}_\xi(\Omega)$ denote the generalized Kato class associated with $M_\Omega(\cdot,\xi)$ (see Section 2.1 and Definition 2.12 for their definitions). We assume

(A1) f is nonnegative and Borel measurable on $\Omega \times [0,\infty)$,
(A2) there is a nonnegative Borel measurable function ψ on $\Omega \times [0,\infty)$ such that
- for each $x \in \Omega$, $\psi(x,\cdot)$ is nondecreasing on $[0,\infty)$ and $\lim_{u\to 0+} \psi(x,u) = 0$,
- $\psi(\cdot, M_\Omega(\cdot,\xi)) \in \mathcal{K}_\xi(\Omega)$,
- whenever $0 \leq u_1 \leq u_2$, we have

$$|f(x,u_1) - f(x,u_2)| \leq \psi(x,u_2)|u_1 - u_2| \quad \text{for all } x \in \Omega.$$

Our main result is as follows.