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Global existence and nonexistence of the initial–boundary value problem for the dissipative Boussinesq equation☆

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ABSTRACT

A class of multidimensional dissipative Boussinesq equations is considered. Under three different cases of initial energy, we prove the existence and uniqueness of global solutions in the energy space and provide sufficient conditions of finite time blow up solutions. The sharp result is given as $E(0) < d$. Particular attention is paid to initial energy at supercritical initial energy ($E(0) > d$) level which is novel contribution of this paper, and new methods will be introduced and some analysis techniques are needed. In addition, the asymptotic behavior of the global solutions is studied.

1. Introduction and main results

In this paper, we focus on the following initial–boundary value problem of the dissipative Boussinesq equation

$$u_{tt} - \Delta u + \Delta^2 u - \alpha\Delta u_t + \gamma\Delta^2 u_t + \Delta f(u) = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+ \tag{1.1}$$

$$u\,|_{\partial\Omega} = 0, \qquad \Delta u\,|_{\partial\Omega} = 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \tag{1.3}$$

where $f(u) = \beta|u|^{p-1}u$, $\beta \in \mathbb{R}, \beta \neq 0$, $1 < p < \infty$, and the constants α, γ satisfy $\gamma \geq 0$ and $\alpha > -\lambda_1\gamma$. Here λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary data on the smooth bounded domain $\Omega \subset \mathbb{R}^n$. The function $u(x,t)$ denotes the unknown function, Δ is the n-dimensional Laplacian, the subscript t indicates the partial derivative with respect to t, and u_0 and u_1 are the given initial value functions.

In [3], starting from Euler's equations, Joseph Boussinesq was the first to derive a partial differential equation with solutions consistent with Russell's observation, which is the existence of solitary waves

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(see [28]). This is nowadays known as the "bad" Boussinesq equation,

$$u_{tt} - u_{xx} - u_{xxxx} + (u^2)_{xx} = 0,$$

since the corresponding initial value problem is not well-posed: the small amplitude ($\varepsilon \ll 1$) family of solutions

$$u(x,t) = \varepsilon e^{ikx - iwt}, \qquad w^2 = k^2 - k^4, \quad k \in \mathbb{R},$$

grows exponentially with time for $|k| > 1$. However, flipping a sign gives a well-posed initial value problem for the "good" Boussinesq equation:

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0.$$

A generalization of one of the Boussinesq type equations is considered which arises in the modeling of nonlinear strings, namely,

$$u_{tt} - u_{xx} + (u_{xx} + f(u))_{xx} = 0. \tag{1.4}$$

It has been proposed in [1] that certain solitary-wave solutions of (1.4) are nonlinearly stable for a range of their wave speeds.

So far, there have been a number of contributions on the "good" Boussinesq equation, its generalized form (1.4) and the damped Boussinesq equation (1.5). In [1], Bona and Sachs obtained some sufficient conditions for the initial data to evolve into a global solution of the equation. Linares, in [19], discussed the small-data Cauchy problem of (1.4). Linares and Scialom considered the asymptotic behavior of solutions for (1.4) with $f(u) = |u|^\alpha u$ ($\alpha > 1$) in [20]. The existence and nonexistence of global solutions to the generalized Boussinesq equation (1.4) was investigated by Liu in [23]. In [14], Kishimoto proved that the initial value problem of "good" Boussinesq equation is locally well-posed in $H^{-\frac{1}{2}}(\mathbb{T})$ and ill-posed in $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$. By employing Besov space $B^s_{p,2}$ with $1 < p < 2$ and positive regularity ($s > 0$), Cho and Ozawa [4] established the global existence and the scattering of a small amplitude solution to the Cauchy problem of the multidimensional version of Eq. (1.4) and their results improved the ones obtained in [24, 36]. Later, some results of [4] were improved by Ferreira [9], in which, the author studied the initial value problem for the n-dimensional ($n \geq 1$) generalized Boussinesq equation (1.4) and proved existence of local and global solutions with singular initial data in weak-L^p spaces. Moreover, long time behavior results and a scattering theory were also obtained.

Boussinesq equation presents an appropriate balance between the nonlinearity and the dispersion because of the existence of solitary-wave solutions, see [2]. However, dissipation is naturally introduced in fluid dynamics through viscosity processes. Counting the viscosity term, the good Boussinesq equation has been modified as the damped Boussinesq equation

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - 2b u_{xxt} = \beta (u^2)_{xx}. \tag{1.5}$$

The existence, uniqueness and long-time asymptotic of solutions to the Cauchy problem and the initial boundary value problem of Eq. (1.5) and its multi-dimensional version has been studied by Varlamov, and we refer to [5,30–33,35,34].

In order to elucidate the interplay between the nonlinearity on one hand, and dispersion, energy input and dissipation on the other, the prototypical Boussinesq-type of equation with dissipation and energy input then reads

$$u_{tt} = \left[\gamma^2 u - \frac{\alpha}{2} u^2 - \beta\gamma u_{xx} - \alpha_4 u_{xxt} - \alpha_2 u_t\right]_{xx} \tag{1.6}$$

which encompasses the oscillations of elastic beams (see [8,6,7]). Here α is the amplitude coefficient, γ is the phase speed of the small disturbances, $(\beta\gamma)$ is the dispersion coefficient, α_2 is the coefficient of