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Uniqueness of weak solutions to a high dimensional Keller–Segel equation with degenerate diffusion and nonlocal aggregation[☆]

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A B S T R A C T

This paper considers weak solutions to the degenerate Keller–Segel equation with nonlocal aggregation: $u_t = \Delta u^m - \nabla \cdot (uB(u))$ in $\mathbb{R}^d \times \mathbb{R}^+$, where $B(u) = \nabla((-\Delta)^{-\frac{\beta}{2}}u)$, $d \geq 3$, $\beta \in [2,d)$, $1 < m < 2 - \frac{\beta}{d}$. In a previous paper of the authors (Hong et al., 2015), a criterion was established for global existence versus finite time blow-up of weak solutions to the problem. A natural question is whether the uniqueness is true for the weak solutions obtained. A positive answer is given in this paper that the global weak solutions must be unique provided the second moment of initial data is finite, which means that the weak solutions are weak entropy solutions in fact. The framework of the proof is based on the optimal transportation method.

1. Introduction

This paper considers the degenerate Keller–Segel equation with nonlocal aggregation

$$u_t - \Delta u^m + \nabla \cdot (uB(u)) = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \tag{1.1}$$

where $d \geq 3$, $B(u) = \nabla((-\Delta)^{-\frac{\beta}{2}}u)$ with $\beta \in [2,d)$, $1 < m < 2 - \frac{\beta}{d}$. By using the Riesz kernel $I_\beta(x) = \frac{1}{\gamma(d,\beta)}|x|^{\beta-d}$ with $\gamma(d,\beta) = \pi^{\frac{d}{2}} 2^\beta \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{d-\beta}{2})}$ [25], Eq. (1.1) can be rewritten as the form of

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u\nabla c), & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ c = I_\beta * u = \dfrac{1}{\gamma(d,\beta)} \dfrac{1}{|x|^\nu} * u, & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \end{cases} \tag{1.2}$$

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where $d \geq 3$, $\nu = d - \beta$, $0 < \nu < d - 2$, $m \in (1, \frac{d+\nu}{d})$, with $u = u(x,t)$ and $c = c(x,t)$ representing the bacteria density and the chemical substance concentration respectively. The Riesz kernel with $\beta = 2$ just is the Newtonian kernel $I_2(x) = \frac{1}{\gamma(d,2)}|x|^{2-d} = \frac{1}{(d-2)}\frac{1}{\omega_d}|x|^{2-d}$, related to the classical potential of the standard K–S model, where ω_d is the measure of the unit sphere in $\mathbb{R}^d$. The more general potential c in the model (1.2) describes the nonlocal aggregation in (1.1). In addition, subject the initial data

$$u|_{t=0} = u_0 \in L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \quad \text{with } \|u_0\|_{L^{p^*}} < C_{d,m} \tag{1.3}$$

to (1.2), where $p^* = \frac{d(2-m)}{d-\nu}$, and $C_{d,m}$ is the universal constant introduced to the existence of global weak solutions [17, Theorem 3.1]. Here and throughout the paper, we use $\|\cdot\|_{L^p}$ to represent $\|\cdot\|_{L^p(\mathbb{R}^d)}$ for simplicity, $p \in [1,\infty]$.

Due to the degeneracy of (1.2), we have to deal with weak solutions. In the previous paper [17], we established a criterion for global existence versus finite time blow-up of weak solutions to the problem (1.2) that the weak solutions must be global if $\|u_0\|_{L^{p^*}} < C_{d,m}$ (required by (1.3)), and would blow up in finite time provided $\|u_0\|_{L^{p^*}}$ above some $\tilde{C}_{d,m} \geq C_{d,m}$. A natural question is whether the uniqueness is true for the weak solutions of (1.2)–(1.3)? The purpose of the present paper is to give a positive answer to this question with $\nu \in ((\frac{d}{2} - 2)_+, d-2)$.

Recall the known studies for the K–S systems

$$\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla c), & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \epsilon \partial_t c = \Delta c + u - \alpha c, & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+. \end{cases} \tag{1.4}$$

The global existence and non-existence of solutions to (1.4) have been greatly studied with many significant results [2–8,11,13,14,23,24,26,28,29], where the involved solutions may be classical or weak ones, depending on $m = 1$ or $m \neq 1$ there. However, much less can be found for the uniqueness of solutions. Among them the uniqueness was proved to (1.4) with $m = 1$ and dimension $d = 2$ in [12,15,16,22]. Egana and Mischler [15] considered (1.4) with $\epsilon = \alpha = 0$. They used a DiPerna–Lions renormalizing argument to get the "optimal regularity" as well as an estimate of the difference of two possible solutions in the critical $L^{\frac{4}{3}}$-norm, and thereby proved the uniqueness of the "free energy" solutions on the maximal existence interval $[0, T^*)$. Corrias, Escobedo, and Matos [12] considered (1.4) with $\epsilon > 0$, $\alpha = 0$. By analyzing the profiles of solutions, they obtained for any $\epsilon > 0$ that the positive integrable and rapidly decaying self-similar solution (u_M, c_M) is unique, with the initial mass M less than some positive constant $M_\epsilon \in [4\pi, 8\pi]$, and $M_\epsilon = 8\pi$ if $\epsilon \in (0, \frac{1}{2}]$. The uniqueness of (u_M, c_M) with small initial data was at first proved for the case of $\epsilon = 1$ in [16,22].

Now consider the nonlinear diffusion situation with $m \neq 1$. For the dimension $d = 1$ Sugiyama [27] proved the uniqueness of weak solutions to (1.4) with the regularity $\partial_t u \in L^1_{\mathrm{loc}}(\mathbb{R} \times (0,T))$, $\epsilon = 0$, $m > 1$. Moreover, when the dimension $d > 1$, $m > \max\{\frac{1}{2} - \frac{1}{d}, 0\}$, Sugiyama et al. established the uniqueness of weak solutions in spaces of Hölder continuous functions by the duality method coupled with the vanishing viscosity method to (1.4) with $\epsilon = 1$ [21], and for $\epsilon = 0$ as well [18], although the existence of weak solutions in that spaces is still unknown.

Recently, the method of optimal transportation is used to prove the uniqueness. Loeper [20] used the method to prove the uniqueness of weak solutions to Vlasov–Poisson system. Carrillo et al. [9,10] established the uniqueness of bounded solutions of (1.4) with $m = 1$, $\epsilon = 0, 1$. Liu and Wang [19] considered (1.4) with $m > 1$, $\epsilon = \alpha = 0$, and obtained the uniqueness of weak solutions.

In the present paper, by using the method of optimal transportation above, we will establish the uniqueness of weak solutions to (1.2)–(1.3) with $d \geq 3$, $m \in (1, \frac{d+\nu}{d})$.