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Instability and stability properties of traveling waves for the double dispersion equation

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ABSTRACT

In this article we are concerned with the instability and stability properties of traveling wave solutions of the double dispersion equation $u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} = -(|u|^{p-1}u)_{xx}$ for $p > 1$, $a > b > 0$. The main characteristic of this equation is the existence of two sources of dispersion, characterized by the terms u_{xxxx} and u_{xxtt}. We obtain an explicit condition in terms of a, b and p on wave velocities ensuring that traveling wave solutions of the double dispersion equation are strongly unstable by blow up. In the special case of the Boussinesq equation ($b = 0$), our condition reduces to the one given in the literature. For the double dispersion equation, we also investigate orbital stability of traveling waves by considering the convexity of a scalar function. We provide analytical as well as numerical results on the variation of the stability region of wave velocities with a, b and p and then state explicitly the conditions under which the traveling waves are orbitally stable.

1. Introduction

The present paper is concerned with the instability and stability properties of traveling wave solutions for the double dispersion equation

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} = -(|u|^{p-1}u)_{xx}, \tag{1.1}$$

where a, b are positive real constants with $a > b$, and $p > 1$. In particular we prove that traveling wave solutions are unstable by blow-up if the wave velocities of the traveling waves are less than a critical wave velocity. We also state explicitly a set of conditions on a, b and p for which the traveling waves are orbitally stable.

The double dispersion equation (1.1) was derived as a mathematical model of the propagation of dispersive waves in a wide variety of situations, see for instance [15,14] and the references therein. Well posedness (and

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related properties) of the Cauchy problem for the double dispersion equation have been studied in the literature by several authors [18,19,8]. It is interesting to note that (1.1) is a special case of the general class of nonlinear nonlocal wave equations

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \tag{1.2}$$

with pseudo-differential operators L and B, studied in [1,4,5]. Indeed, for the case

$$L = (I - aD_x^2)(I - bD_x^2)^{-1}, \qquad B = (I - bD_x^2)^{-1}, \qquad g(u) = -|u|^{p-1}u, \tag{1.3}$$

where I is the identity operator and D_x denotes the partial derivative with respect to x, (1.2) reduces to (1.1). The well-posedness of the Cauchy problem for the general class (1.2) was studied in [1] and then the parameter dependent thresholds for global existence versus blow-up were established in [4] for power nonlinearities. In a recent study [5] on (1.2), again for power nonlinearities, the existence of traveling wave solutions $u = \phi_c(x - ct)$, where $c \in \mathbb{R}$ is the wave velocity, has been established and orbital stability of the traveling waves has been studied. The orbital stability is based on the convexity of a certain function $d(c)$ related to conserved quantities. Furthermore, it has been shown that when $L = I$, (1.2) becomes a special case of the Klein–Gordon-type equations and $d(c)$ can be computed explicitly. In [5] the sharp threshold of instability/stability of traveling waves for this regularized Klein–Gordon equation has been established. In other words, for $L = I$, it has been shown that traveling wave solutions of (1.2) are orbitally stable for

$$\frac{p-1}{p+3} < c^2 < 1 \tag{1.4}$$

and are unstable by blow-up for

$$c^2 < \frac{p-1}{p+3}. \tag{1.5}$$

It remains an open question, however, whether a sharp threshold of instability/stability can be obtained for the double dispersion equation (1.1) which is another special case of (1.2).

For some limiting cases of (1.1), the above question was fully answered in the literature. For the special case $a = 1$, $b = 0$; (1.1) becomes the (generalized) Boussinesq equation [3]

$$u_{tt} - u_{xx} + u_{xxxx} = -(|u|^{p-1}u)_{xx} \tag{1.6}$$

which has received much attention in the literature. It was established in [2] that solitary wave solutions of (1.6) are orbitally stable if

$$\frac{p-1}{4} < c^2 < 1 \quad \text{and} \quad 1 < p < 5. \tag{1.7}$$

In [11], it was proved that solitary waves for (1.6) are orbitally unstable if

$$c^2 < \frac{p-1}{4} \quad \text{and} \quad 1 < p < 5, \tag{1.8}$$

or

$$c^2 < 1 \quad \text{and} \quad p \geq 5. \tag{1.9}$$

On the other hand, in [12] it was shown that traveling wave solutions of (1.6) are strongly unstable by blow-up for

$$c^2 < \frac{p-1}{2(p+1)}. \tag{1.10}$$

In the limiting case $a = b$; (1.1) reduces to

$$u_{tt} - u_{xx} = -(1 - bD_x^2)^{-1}(|u|^{p-1}u)_{xx}, \tag{1.11}$$