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Global wellposedness of cubic Camassa–Holm equations

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ABSTRACT

In this paper, we study the Cauchy problem of cubic Camassa–Holm equation. This equation is derived from a model of shallow water dynamics. We prove the global existence of entropy weak solution to this problem in space H^1 with its derivative in BV. The stability and uniqueness of entropy weak solution are obtained in $W^{1,1}$.

1. Introduction

We consider the cubic Camassa–Holm equation (cCH):

$$m_t + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx}, \tag{1.1}$$

or

$$u_t - u_{xxt} + 3u^2 u_x - u_x^3 - 4uu_x u_{xx} + 2u_x u_{xx}^2 + (u_x^2 - u^2)u_{xxx} = 0. \tag{1.2}$$

If written as a first-order equation of u, it is

$$u_t + u^2 u_x = \frac{1}{3}u_x^3 - \Phi(u), \tag{1.3}$$

with

$$\Phi(u) = (1 - \partial_x^2)^{-1}\left[\frac{1}{3}u_x^3\right] + \partial_x(1 - \partial_x^2)^{-1}\left[\frac{2}{3}u^3 + uu_x^2\right]. \tag{1.4}$$

The initial data is

$$u|_{t=0} = u_0(x). \tag{1.5}$$

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This equation is a complete integrable water wave equation proposed by several different authors [13,20,21]. It is a suitable approximation of incompressible Euler system without swirl. For smooth solutions to the equation, we have the following two conserved quantities:

$$H_1(u) = \int_{\mathbb{R}} (u^2 + u_x^2)dx, \qquad H_2(u) = \int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4 \right) dx. \tag{1.6}$$

In fact, these two invariant quantities are the Hamiltonian functionals of (cCH), with the compatible Hamiltonian operators

$$\mathcal{J}_2 = -\partial_x m \partial_x^{-1} m \partial_x, \qquad \mathcal{J}_1 = -\frac{1}{4}(\partial_x - \partial_x^3). \tag{1.7}$$

With these notations, the equation can be written in the bi-Hamiltonian form

$$m_t = \mathcal{J}_2 \frac{\delta H_1}{\delta m} = \mathcal{J}_1 \frac{\delta H_2}{\delta m}. \tag{1.8}$$

In recent decades, the partial differential equations from the fields of integrable systems have been studied adequately. Among them are KdV equation (KdV), modified KdV equations (mKdV), generalized KdV equations (gKdV), Camassa–Holm equations (CH), generalized Camassa–Holm equations (gCH), Degasperis–Procesi equation (DP), and so on [7,19]. Some of them are completely integrable. They have Lax pair formulation of the equations, which allows to apply inverse scattering techniques [6,9,10,12]. However, not all equations are completely integrable. From analytical point of view, the Hamiltonian quantities usually provide us the functional spaces for the solutions, which is not sufficient to obtain a solution. It usually requires further regularity to guarantee compactness. For example, (gKdV) is a semilinear dispersive equation. In [17], the authors proved global well-posedness by using contraction mapping theorem via dispersive estimates.

Camassa–Holm equations are closely related with KdV equations in the sense that Camassa–Holm equations are the tri-Hamiltonian duality of KdV equations [20]. However, unlike KdV equations, Camassa–Holm equations do not have that strong dispersive effect. In fact, they are more like quasilinear hyperbolic equations. They have the wave breaking phenomena [8] which is not shared by the KdV-type equations. The problem of global well-posedness of Camassa–Holm equation is equivalent to the problem of how to continue the solution after the singularity appears. There are two methods to deal with it in general: compactness method and method of characteristics. Global existence of weak solutions for (CH) is proved by compensated compactness method in [11,22], and a result of weak–strong uniqueness is proved in [23]. For (DP), global wellposedness, including existence, uniqueness and stability, is proved in [5]. The other way to obtain global well-posedness for (CH) is by the method of characteristics. In [2,3], conservative and dispersive weak solutions are constructed, and in [1], the authors proved the uniqueness of conservative weak solutions.

Different from the Camassa–Holm equation, the cubic Camassa–Holm equation has higher nonlinearity. This difficulty is manifested in the equation of characteristics:

$$\frac{dx}{dt} = (u^2 - u_x^2)(t, x(t)). \tag{1.9}$$

From the two conserved Hamiltonians, it is hard to expect the right-hand-side to be continuous with respect to x. So this characteristic ODE may not be globally wellposed. The method of characteristics does not apply here.

Here I would like to mention another interesting equation—Novikov equation

$$u_t - u_{xxt} + 4u^2u_x - 3uu_xu_{xx} - u^2u_{xxx} = 0. \tag{1.10}$$

This equation shares the same conserved quantity with (cCH) but has lower order of nonlinearity. An advantage of this is reflected on the equation of characteristics:

$$\frac{dx}{dt} = u^2(t, x(t)). \tag{1.11}$$