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Characterization of stadium-like domains via boundary value problems for the infinity Laplacian

Graziano Crasta[a,*], Ilaria Fragalà[b]

[a] *Dipartimento di Matematica "G. Castelnuovo", Univ. di Roma I, P.le A. Moro 2 – 00185, Roma, Italy*
[b] *Dipartimento di Matematica, Politecnico, Piazza Leonardo da Vinci, 32–20133, Milano, Italy*

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A B S T R A C T

We give a complete characterization, as "stadium-like domains", of convex subsets Ω of $\mathbb{R}^n$ where a solution exists to Serrin-type overdetermined boundary value problems in which the operator is either the infinity Laplacian or its normalized version. In case of the not-normalized operator, our results extend those obtained in a previous work (where the problem was solved under some geometrical restrictions on Ω), while in the case of the normalized operator they are new. In the normalized case, we also show that stadium-like domains are precisely the unique convex sets in $\mathbb{R}^n$ where the solution to a Dirichlet problem is of class $C^{1,1}(\Omega)$.

1. Introduction

Consider the following Serrin-type problems for the infinity Laplace operator Δ_∞ or its normalized version Δ_∞^N:

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = c & \text{on } \partial\Omega, \end{cases} \tag{1}$$

* Corresponding author.
E-mail addresses: crasta@mat.uniroma1.it (G. Crasta), ilaria.fragala@polimi.it (I. Fragalà).

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and

$$\begin{cases} -\Delta_\infty^N u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = c & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The aim of this paper is to provide a complete characterization of convex domains $\Omega \subset \mathbb{R}^n$ where such problems admit a solution.

Following the seminal paper by Serrin [35] and the huge amount of literature after it (see for instance [5,6,18–21,28,36]), overdetermined boundary value problems involving the infinity Laplace operator were firstly considered only few years ago by Buttazzo and Kawohl (see [7]). In fact, due to the high degeneracy of the operator, all the different methods exploited in the literature to obtain symmetry results for overdetermined boundary value problems fail when applied to problems (1)–(2).

In [7], Buttazzo and Kawohl dealt with a simplified version of problems (1)–(2), which consists in looking for solutions having the same level lines as the distance function to the boundary of Ω, which are called *web-functions* (see Section 2). This simplification essentially reduces the problem to a one-dimensional setting, allowing to prove that the existence of a web-solution implies a precise geometric condition on Ω, which is the coincidence of its *cut locus* and *high ridge* (see again Section 2 for the definitions). In particular, such condition does not imply symmetry, at least if taken alone without any additional boundary regularity requirement. In our previous paper [13] we studied the geometry of domains whose cut locus and high ridge agree, by providing a complete characterization of them in dimension $n = 2$, and in higher dimensions for convex domains; in particular, these results reveal that planar convex sets with the same cut locus and high ridge are tubular neighborhoods of a line segment (possibly degenerated into a point). Moreover, in [12] we were able to carry over the study of problem (1) in the class of web-functions, by dropping all the regularity hypotheses on both the domain and the solution previously asked in [7].

The study of problems (1)–(2) in their full generality, namely without imposing the solution to be a web-function, turns out to be much more challenging. As for problem (2), to our knowledge it has never been undergone. As for problem (1), in a recent work we proved that, among convex sets, those having the same cut locus and high ridge – that we call "stadium-like domains" – are the only ones for which *whatever* solution (not necessarily of web type) exists, see [14, Thm. 5]. As a drawback, we needed to ask the following *a priori* geometrical hypothesis on the convex domain Ω: there exists an inner ball, of radius equal to the maximum of the distance from the boundary, touching $\partial\Omega$ at two diametral points. Moreover, we also needed the technical assumption that Ω satisfies an interior sphere condition at every point of the boundary.

The approach we adopted for the proof relies on the study of a suitable P-function along the gradient flow of the unique solution to the Dirichlet problem. In particular, the diametral ball condition was used as a fundamental picklock to get the result. Indeed, it allowed us to overcome the possible lack of regularity of the solution, which is an intrinsic phenomenon; we refer to [14, Sections 5 and 6] for more details, including regularity thresholds.

However, there was no reason to think that the geometric assumptions made on Ω should be really necessary, so that the conclusion reached in [14] was not completely satisfactory.

We can now introduce the contents of this paper, by describing its main results:

- Theorem 2 improves the achievement of [14, Thm. 5], by showing that it continues to hold without any geometric assumptions on Ω except convexity, i.e. when both the diametral ball assumption and the interior sphere condition are removed. Contrarily to our previous belief, it is possible to arrive at this conclusion by completely circumventing regularity matters, but rather exploiting the observation that a suitable web-function is always a super-solution to our problem (see Proposition 12).