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On the stability of the cut locus

Paolo Albano

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40127 Bologna, Italy

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ABSTRACT

In $\mathbb{R}^d$ we consider a Riemannian metric, g, and an open bounded subset, Ω. We study the stability of the cut locus associated with Ω and g w.r.t. perturbations both of the set Ω and of the metric g. In order to have the stability of the cut locus, we assume C^2 regularity of the data, the metrics and the sets (in the case of sets with $C^{1,1}$ boundaries, the cut locus may be unstable). We prove that to C^2 perturbations both of the set and of the metric correspond small changes of the cut locus w.r.t. the Hausdorff distance, i.e. the cut locus is stable in the C^2 category. We give some examples showing that C^1 perturbations may lead to large variations of the cut locus.

1. Introduction and statement of the result

Let Ω' be an open bounded subset of $\mathbb{R}^d$. Set

$$\mathcal{A}(\Omega') = \{\Omega \subset\subset \Omega' \mid \Omega \text{ is an open set with boundary of class } C^2\}$$

and

$$\mathcal{G}(\Omega') = \{g \mid g \text{ is a Riemannian metric with } C^2 \text{ coefficients defined in } \Omega'\}.$$

Let $\Omega \in \mathcal{A}(\Omega')$ and let $g \in \mathcal{G}(\Omega')$. By using the metric g, given a piecewise differentiable curve on $\overline{\Omega}$, $\gamma : [0, 1] \longrightarrow \overline{\Omega}$, we can measure its length,

$$L(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t))}\, dt.$$

Hence, for every couple of points $x, y \in \Omega$, we denote by $d(x, y)$ the distance between x and y, i.e.

$$d(x, y) = \inf L(\gamma)$$

E-mail address: paolo.albano@unibo.it.

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(the infimum is taken in the set of all the piecewise differentiable curves on $\overline{\Omega}$ connecting x with y). We consider the distance function from the boundary of Ω, $\partial\Omega$, defined as

$$d_{\partial\Omega}(x) = \inf_{y\in\partial\Omega} d(x,y), \quad (x \in \Omega).$$

Let γ be a geodesic starting at a point of $\partial\Omega$. We say that a point of γ, z, is a *cut point* of $\partial\Omega$ (along γ) if it is the first point on γ such that for any point y in γ beyond z, there exists a geodesic from a point of $\partial\Omega$ to y shorter than γ. We denote by $\mathrm{cut}(\Omega, g)$ the set of all the cut points of $\partial\Omega$ (in Ω) associated with the Riemannian metric g.

We study the stability of the cut locus w.r.t. perturbations both of the set Ω and of the metric g.

We recall some basic properties of the cut locus:

(1) the cut locus has measure zero[1] (see e.g. [14] for general and precise rectifiability results).
(2) The C^2 regularity of the Riemannian metric and of the boundary implies that the cut locus is a closed set (for a discussion of such a property we refer the reader to Remark 2.1 in the sequel).
(3) The cut locus, $\mathrm{cut}(\Omega, g)$, has the same homotopy type as the set Ω (see e.g. [4]).
(4) The cut locus stays away from the boundary of the open set under exam, $\partial\Omega$, (see e.g. [1]).

In particular, $\mathrm{cut}(\Omega, g) \subset \Omega$ is a compact set, for every $\Omega \in \mathcal{A}(\Omega')$ and for every $g \in \mathcal{G}(\Omega')$.

We measure the variations of the cut locus by using the Hausdorff distance between compact sets: given two compact sets $K, L \subset \Omega'$ we define

$$d_H(K,L) = \max\{\max_{x\in K} d^e_L(x), \max_{L} d^e_K(x)\}.$$

Hereafter $d^e_K(x)$ denotes the Euclidean distance function of x from the set K.

For every $g \in \mathcal{G}(\Omega')$ we have

$$g_x(\xi) = \langle G(x)\xi, \xi\rangle, \quad x \in \Omega', \ \xi \in \mathbb{R}^d. \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle$ stands for the standard Euclidean product in $\mathbb{R}^d$ and $G(\cdot)$ is a family of positive definite matrices, with entries of class C^2.

Definition 1.1 (*Convergence of the Metrics and of the Sets*).

(i) For $k = 1, 2$, we say that the sequence $g_j \in \mathcal{G}(\Omega')$ converges to $g \in \mathcal{G}(\Omega')$ in C^k if

$$\lim_{j\to\infty} \sum_{|\alpha|\le k} \sup_{x\in\Omega'} \|\partial^\alpha_x (G_j - G)(x)\| = 0.$$

(Here $\|\cdot\|$ stands for a norm in the space of the positive definite $d \times d$ matrices.)

(ii) For $k = 1, 2$, we say that the sequence $\Omega_j \in \mathcal{A}(\Omega')$ converges to $\Omega \in \mathcal{A}(\Omega')$ in C^k if
  (1) there exists an open neighbourhood of $\partial\Omega$, $W \subset \Omega'$, such that $\partial\Omega_j \subset W$, $j = 1, 2, \ldots$.
  (2) Denoting by $\mathrm{dist}(x, \partial\Omega_j) = d^e_{\Omega'\setminus\Omega_j}(x) - d^e_{\Omega_j}(x)$ the (Euclidean) signed distance of x from $\partial\Omega_j$, $\mathrm{dist}(\cdot, \partial\Omega)$ and $\mathrm{dist}(\cdot, \partial\Omega_j)$ are in $C^k(W)$, $j = 1, 2, \ldots$.
  (3) $\lim_{j\to\infty} \sum_{|\alpha|\le k} \sup_{x\in W} |\partial^\alpha_x(\mathrm{dist}(x, \Omega_j) - \mathrm{dist}(x, \Omega))| = 0$.

[1] This property holds under weaker regularity conditions on the boundary of Ω: the cut locus associated with a closed set $C \subset \mathbb{R}^d$ has measure zero (see [5]).