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Nonlocal problems at nearly critical growth

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ABSTRACT

We study the asymptotic behavior of solutions to the nonlocal nonlinear equation $(-\Delta_p)^s u = |u|^{q-2}u$ in a bounded domain $\Omega \subset \mathbb{R}^N$ as q approaches the critical Sobolev exponent $p^* = Np/(N - ps)$. We prove that ground state solutions concentrate at a single point $\bar{x} \in \overline{\Omega}$ and analyze the asymptotic behavior for sequences of solutions at higher energy levels. In the semi-linear case $p = 2$, we prove that for smooth domains the concentration point $\bar{x}$ cannot lie on the boundary, and identify its location in the case of annular domains.

1. Introduction and main results

Let Ω be a smooth bounded domain of $\mathbb{R}^N, N > ps, s \in (0, 1)$ and $p > 1$. For any sufficiently small $\varepsilon > 0$, we consider the nonlocal nonlinear problem

$$\begin{cases} (-\Delta_p)^s u = |u|^{p^*-2-\varepsilon} u, & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $p^* = Np/(N - sp)$ is the critical exponent for the immersion of

$$W_0^{s,p}(\Omega) := \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx < +\infty,\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}$$

into the space $L^q(\Omega)$. By a weak solution to problem (1.1) we mean a critical point for the C^1 functional $I_\varepsilon : W_0^{s,p}(\Omega) \to \mathbb{R}$ defined by

$$I_\varepsilon(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dx\, dy - \frac{1}{p^* - \varepsilon} \int_\Omega |u|^{p^*-\varepsilon} dx.$$

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The nonlinear operator $(-\Delta_p)^s : W_0^{s,p}(\Omega) \to W^{-s,p}(\Omega)$ is defined (up to a multiplicative constant which we will ignore in the following) as the differential of the first term in I_ε and it can be represented, on smooth functions, by

$$(-\Delta_p)^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N.$$

Solutions of (1.1) inherit some mild smoothness when seen as general non-homogeneous non-local equations. The regularity theory for $p \neq 2$ is far from complete, however the basic continuity instances of it are covered at the interior in [9,8] for the homogeneous case and in [18,5] for nonhomogeneous equations, while in [16] regularity up to the boundary is established for boundedly nonhomogeneous equations.

In this paper, we are interested in the asymptotic behavior of a sequence of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to (1.1) as $\varepsilon \searrow 0$, as determined by the limit energy $c = \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon)$.

The interest in such "nearly critical" problems arises from the fact that for $\varepsilon > 0$ compactness is recovered and the problem is more easily solved, hopefully providing in the limit a solution to the non-compact problem at $\varepsilon = 0$. In many cases, however, the validity of a Pohozaev identity rules out existence of nontrivial solutions for $\varepsilon = 0$, and the asymptotic behavior of the approximating solutions describes the phenomenon of lack of solutions in the limit.

In the seminal paper [1], the asymptotic behavior of the (unique and radial) solution u_ε

$$\begin{cases} -\Delta u_\varepsilon = u_\varepsilon^{2^*-\varepsilon}, \quad u_\varepsilon > 0 & \text{in } B, \\ u_\varepsilon = 0 & \text{on } \partial B \end{cases} \quad \text{where } B \text{ is a ball in } \mathbb{R}^3 \tag{1.2}$$

is considered, showing, among other things, that u_ε concentrates at a single point, the center of B, at a rate $\max u_\varepsilon = u_\varepsilon(0) \simeq \sqrt{\varepsilon}$. This kind of results were extended and refined in [7]. For general smooth domains, where uniqueness of solutions (and nonexistence of the latter for $\varepsilon = 0$) to (1.2) is lost, the same kind of behavior is proved in [14,26] for the *ground states* of (1.2), namely, nontrivial solutions minimizing the associated energy functional. Indeed, regardless of the existence of positive solutions of the limiting equation, ground states *always* concentrate *all* their mass at some point, which is therefore called the point of concentration. Through a rather fine analysis, the concentration point is shown in [25] to be a minimum point of the Robin function of Ω. For smooth domains, this implies that the concentration points cannot belong to $\partial\Omega$, while for nonsmooth domain the boundary concentration phenomenon can happen, as shown e.g. in [11].

For more general, nonlinear equations, the situation is less clear. In [13] the concentration of ground states is proved for the p-Laplacian via critical points methods, while in [22] via Γ-convergence ones (see the latter for more references on this approach). In [21] more general and non regular operators are considered. However, the location of concentration points for ground states is not clear, (even trying to prove that they do not belong to $\partial\Omega$ in smooth domains), and precise asymptotic behavior of the maxima are even less so. It is worth noting, however, that for a different but related problem involving the p-Laplacian, the location of concentration points has been determined with the technique of p-harmonic transplantation, see [10].

Regarding the nonlocal problem (1.1), the semi-linear case $p = 2$ is considered in [24] with a Γ-convergence approach and in [23] via profile decomposition. The latter approach relies on the Hilbert structure to take advantage of abstract profile decomposition theory, but, as shown in [17], no such precise decomposition can hold *for general bounded sequences* when $p \neq 2$. A more suitable profile decomposition when $p \neq 2$ has recently been obtained *for Palais–Smale sequences* in [6], which in principle may lead to the same kind of results we will discuss in a short while. However, a direct approach through non local Concentration-Compactness seems more convenient for ground state solutions, and is flexible enough to provide informations at higher critical levels as well.