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Global solutions to the generalized Leray-alpha equation with mixed dissipation terms

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ABSTRACT

Due to the intractability of the Navier–Stokes equation, it is common to study approximating equations. Two of the most common of these are the Leray-α equation (which replaces the solution u with $(1-\alpha^2\mathcal{L}_1)u$ for a Fourier Multiplier $\mathcal{L}$) and the generalized Navier–Stokes equation (which replaces the viscosity term $\nu\triangle$ with $\nu\mathcal{L}_2$). In this paper we consider the combination of these two equations, called the generalized Leray-α equation. We provide a brief outline of the typical strategies used to solve such equations, and prove, with initial data in a low-regularity $L^p(\mathbb{R}^n)$ based Sobolev space, the existence of a unique local solution with $\gamma_1+\gamma_2>n/p+1$. In the $p=2$ case, the local solution is extended to a global solution, improving on previously known results.

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1. Introduction

The incompressible form of the Navier–Stokes equation is given by

$$\begin{aligned} &\partial_t u + (u\cdot\nabla)u = \nu\triangle u - \nabla p, \\ &u(0,x) = u_0(x), \quad \text{div }(u) = 0 \end{aligned} \tag{1.1}$$

where $u: I\times\mathbb{R}^n\to\mathbb{R}^n$ for some time strip $I=[0,T), \nu>0$ is a constant due to the viscosity of the fluid, $p: I\times\mathbb{R}^n\to\mathbb{R}^n$ denotes the fluid pressure, and the initial data $u_0:\mathbb{R}^n\to\mathbb{R}^n$. The requisite differential operators are defined by $\triangle=\sum_{i=1}^n \frac{\partial^2}{\partial^2_{x_i}}$ and $\nabla=\left(\frac{\partial}{\partial_{x_i}},\dots,\frac{\partial}{\partial_{x_n}}\right)$.

There is a very wide collection of existence results for the Navier–Stokes equation (see [8]). In dimension $n=2$, the existence of a unique (with respect to the initial condition) global solution to the Navier–Stokes equation is well known (see [7]; for a more modern reference, see Chapter 17 of [16]). However, establishing an analogous result for the dimension $n\geq 3$ case has proven to be exceptionally challenging.

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To illustrate this, we recall two classical results. First, in [3], Kato and Fujita proved the existence of a unique (but not global) solution for initial data u_0 in the inhomogeneous Sobolev space $H^{1/2,2}(\mathbb{R}^3)$. One modern approach to obtaining this result is to show that a solution to the Navier–Stokes equation is the fixed point of an operator, and then to apply the Contraction Mapping Principle to guarantee the existence of a (unique) fixed point. This procedure requires functional and harmonic analysis techniques which only apply on short time intervals, and so this technique is rarely able to provide global existence results.

A standard method for obtaining a global existence result is to use energy estimates. For the Navier–Stokes equation, it is well known that the energy of a solution decreases in positive time. That is, if u is a solution to the Navier–Stokes equation, then $\|u(t,\cdot)\|_{L^2}$ is a decreasing function of time. In [9], Leray used this energy estimate to prove that there exists a global, but not necessarily unique, solution to the Navier–Stokes equation with initial data $u_0 \in L^2(\mathbb{R}^3)$.

Given a unique local solution from fixed point method and an energy estimate, a bootstrapping argument can be used to generate a unique global solution. An example of such an argument can be found in Section 5, but generally a bootstrapping argument requires the energy bound to be in the same function space as the initial data in the local existence result. In the context of the current discussion, the fixed point method required initial data in $H^{1/2,2}(\mathbb{R}^3)$, but the energy bound only gives that the L^2 norm of the solution is decreasing. This gap of a half a point of regularity between the unique local existence result and the energy estimate prevents the bootstrap.

In response to the difficulty in closing this gap, many approximating equations have been studied. One of these is the Leray-α equation, which is

$$\begin{aligned} &\partial_t(1-\alpha^2\mathcal{L}_\gamma)u + \nabla_u(1-\alpha^2\mathcal{L}_\gamma)u - \nu\triangle(1-\alpha^2\mathcal{L}_\gamma)u = -\nabla p, \\ &u(0,x) = u_0(x), \qquad \text{div } u_0 = \text{div } u = 0, \end{aligned} \tag{1.2}$$

where we recall that $\nabla_u v = (u\cdot\nabla)v$ and $\mathcal{L}_\gamma$ is a Fourier multiplier with symbol $m(\xi) = -|\xi|^\gamma$. Note that setting $\alpha = 0$ returns the standard Navier–Stokes equation.

The advantage of the Leray-alpha equation over the Navier–Stokes equation is that the Leray-alpha equation has an improved energy bound. Specifically, it is straightforward to show that if u is a solution to (1.2), then $\|u(t,\cdot)\|_{H^{\gamma,2}}$ is a decreasing function of time. In this case, it can be shown that, if $\gamma = 1/2$, then Eq. (1.2) has a unique global solution with initial data $u_0 \in H^{1/2,2}(\mathbb{R}^3)$.

Another option is to generalize the dissipative term $-\nu\triangle u$. This is called the generalized Navier–Stokes equation, and is given by

$$\begin{aligned} &\partial_t u + (u\cdot\nabla)u = \nu\mathcal{L}_\gamma u - \nabla p, \\ &u(0,x) = u_0(x), \qquad \text{div } (u) = 0, \end{aligned}$$

where $\mathcal{L}_\gamma$ is the same operator as above. Choosing $\gamma = 2$ returns the standard Navier–Stokes equation. In this equation, as γ increases, the regularity required of the initial data to obtain a unique local solution decreases, and it is known that this equation has a global solution in dimension three if $\gamma \geq 2.5$ (originally in [11]; see [18] for a more modern reference).

Combining these two approaches returns the generalized Leray-α equation, given by

$$\begin{aligned} &\partial_t(1-\alpha^2\mathcal{L}_2)u + \nabla_u(1-\alpha^2\mathcal{L}_2)u - \nu\mathcal{L}_1(1-\alpha^2\mathcal{L}_2)u = -\nabla p, \\ &u(0,x) = u_0(x), \qquad \text{div } u_0 = \text{div } u = 0, \end{aligned} \tag{1.3}$$

where the operators $\mathcal{L}_i$, with $i = 1, 2$, are Fourier multipliers with symbols $m_i(\xi) = -|\xi|^{\gamma_i}$. Note that setting $\gamma_1 = 2$ and $\gamma_2 = 0$ returns the original Navier–Stokes equation.

These types of generalizations have also been applied to other partial differential equations. For example, in [10], Linshiz and Titi studied various generalizations of the Magnetohydrodynamic (MHD) system and [2], in which Bessaih and Ferrario studied the Boussinesq system with generalized dissipative terms.

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