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Blowup and global solutions in a chemotaxis–growth system

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ABSTRACT

We study a Keller–Segel type of system, which includes growth and death of the chemotactic species and an elliptic equation for the chemo-attractant. The problem is considered in bounded domains with smooth boundary as well as in the whole space. In case the random motion of the chemotactic species is neglected, a hyperbolic–elliptic problem results, for which we characterize blow-up of solutions in finite time and existence of regular solutions globally in time, in dependence on the systems parameters. In this case, convexity of the domain is needed. For the parabolic–elliptic problem in dimensions three and higher, we establish global existence of regular solutions in a limiting case, which is an extension of the results given by Tello and Winkler (2007).

1. Introduction

The classical Keller–Segel model for chemotaxis was introduced in [5] as a model for self-organization of cellular slime-mold amoebae, which produce the chemical they are attracted to themselves when they are under starvation conditions, in order to survive. Therefore the amoebae do not reproduce any more at this stage.

In this paper we additionally take into account growth and death of a cell species which moves chemotactically and produces the chemo-attractant itself. So we look at systems of the form

$$\partial_t u = \varepsilon \Delta u - \nabla \cdot (u \nabla \chi(v)) + f(u, v), \tag{1.1}$$

$$\tau \partial_t v = \eta \Delta v + g(u, v). \tag{1.2}$$

Several questions are of interest in this setting. For $\chi(v) = v$, $f(u, v) = 0$, and $g(u, v) = -\alpha_1 v + \alpha_2 u$ the broadly discussed version of the classical Keller–Segel model for slime mold amoebae appears, which (among

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others) was analyzed w.r.t. the existence of global solutions and finite time blow-up. In two space dimensions the blowup of solutions for this system indicates the possibility of self-organization due to chemotaxis. This final self-organization of the slime mold amoebae is a spatially highly structured 3D phenomenon. Therefore a suitable 2D model which describes the motion and aggregation of the amoebae and at the same time captures the main driving mechanism for the initiation of the final developmental structuring process should, for certain parameter regimes, not show existence of global solutions, in order to be of biological relevance, see [2,4] for $\tau = 0$. In higher dimensions, in difference to the 2D setting, the Keller–Segel system, without modeling any further mechanisms, is more of mathematical interest, unless one considers completely freely moving/flying species/objects which are attracted by an external source.

Another interesting case is taking $\varepsilon = 0$, the limiting case, which neglects diffusion/random motion of the cells completely.

Here we mainly deal with $\tau = 0$, since the diffusion of the chemical molecules is much faster than the motion of the cells, compare [4].

For fixed χ the exact interplay between f and g mainly drives the behavior of the solutions w.r.t. global existence or blowup.

1.1. Known results

Osaki and Yagi in [9] considered the case $\varepsilon > 0, \tau = 1, \eta > 0$, and $g(u,v) = -\alpha_1 v + \alpha_2 u$ in $\mathbb{R}^2 \times (0,\infty)$. For $\sup_{v\geq 0} |d^i\chi(v)/dv^i| < \infty, i = 1,2,3$ and $f(u,v) = f(u)$ with $f(0) = 0$ and $f(u) = (-\mu u + r)u$ for sufficiently large u, where $\mu > 0$ and $-\infty < r < \infty$, existence and uniqueness of global solutions were shown. Typical functionals for χ are $\chi(v) = v, \chi(v) = \log(v+1)$, and $\chi(v) = v/(v+1)$.

Under the same conditions, Osaki, Tsujikawa, Yagi, and Mimura in [8] considered the problem in a spatially two dimensional bounded domain with Neumann boundary conditions. Existence and uniqueness of global solutions were shown, as well as the existence of an attractor to which every solution converges at an exponential rate.

Aida, Osaki, Tsujikawa, Yagi, and Mimura in [1] also considered the problem in a spatially two dimensional bounded domain with Neumann boundary conditions, but allowed for a singular χ at zero, e.g. $\chi(v) = \log v$ or $\chi(v) = 1/v$, but otherwise smooth as before. Additionally it was assumed, that $f'(0) \neq 0$. Existence of a unique global solution was proved and some asymptotics were provided.

Tello and Winkler in [11] considered the system in a bounded domain for arbitrary spatial dimension n with $\varepsilon = 1, \chi(v) = \chi_0 v$ and $f(u,v) = f(u)$ with $f(u) \leq c_1 - \mu u^2, f(u) > 0$ for $0 < u < 1$ and $f(u) < 0$ for $u > 1$. A typical example is $f(u) = \mu u(1-u)$. Further they assumed that $\tau = 0$ and $g(u,v) = u - v$. They proved existence of a unique global bounded solution for either $n \leq 2$ or for $n \geq 3, \mu > (n-2)\chi_0/n$. For all $n \geq 1$ and $\mu > 0$ there exists at least one global weak solution in case $f(u) \geq -c_0(1+u^2)$ for all $u > 0$. Further steady state solutions of the system were discussed.

In [13] Winkler considered a smooth, bounded, convex spatial domain in $\mathbb{R}^n$ for $\tau > 0, \varepsilon = 1, \eta = 1$, and $\chi(v) = \chi_0 v$, with $f(0) \geq 0, f(u,v) = f(u) \leq ru - \mu u^2$, and $g(u,v) = u - v$. For μ sufficiently large and smooth initial data unique global-in-time classical solutions exist, that are bounded.

Winkler in [14] considered a bounded, convex spatial domain in $\mathbb{R}^n$ with $\varepsilon = \eta = \tau = 1, \chi(v) = \chi_0 v, f(u,v) = u - \mu u^2, g(u,v) = u - v$, and Neumann boundary conditions. Then there is an $M > 0$ such that when $\mu/\chi > M$, a global, classical solution exists.

For the spatially one dimensional problem in a bounded open interval, the limiting problem for $\varepsilon = 0$ was discussed by Winkler in [15]. The PDE-system without random motion of the cells was analyzed for $\tau = 0, \eta = 1, \chi(v) = v$, and $f(u,v) = ru - \mu u^2, g(u,v) = u - v$. For $\mu \geq 1$ all solutions for sufficiently regular initial data are global in time. For $\mu < 1$ some solutions blow-up in finite time.