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Steady state solutions for a general activator–inhibitor model

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ABSTRACT

We discuss existence results for a singular Gierer–Meinhardt elliptic system with zero Dirichlet boundary conditions, which originally arose in studies of pattern-formation in biology and has interesting and challenging mathematical properties. The mathematical difficulties are that the system becomes singular near the boundary and it is non-quasimonotone. We show the existence of positive solutions for the general activator–inhibitor model.

1. Introduction

The generalized Gierer–Meinhardt or activator–inhibitor model described by

$$\begin{cases} u_t = D_1\Delta u - \mu u + \dfrac{u^p}{v^q} + \rho(x), \quad x \in \Omega,\ t > 0, \\ v_t = D_2\Delta v - \nu v + \dfrac{u^r}{v^s}, \quad x \in \Omega,\ t > 0, \\ u(x,0) = \phi(x) > 0 \quad \text{and} \quad v(x,0) = \psi(x) > 0, \quad x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

often occurs in the study of biological pattern formation. Here $D_1, D_2, \mu, \nu, p, q, r, s$ are positive and Ω is a bounded domain in R^N with a smooth boundary. u and v represent the concentrations of the activator and inhibitor, respectively, $\rho(x) \geq 0$ (see [8,9]). Such systems are difficult to treat due to the lack of a variational structure or a priori estimates. Although boundary conditions were not explicitly mentioned in the original paper, most works assume a bounded domain and Neumann boundary conditions (see [1,3,13]). Many authors have investigated Dirichlet boundary conditions (see [4,5,7,11]). In this paper, we consider

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the existence of the steady state solutions to (1.1):

$$\begin{cases} \Delta u - \mu u + \dfrac{u^p}{v^q} + \rho(x) = 0, & x \in \Omega, \\ \Delta v - \nu v + \dfrac{u^r}{v^s} = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Choi and McKenna [4] obtained the existence of radially symmetric solutions in the case $\Omega = (0,1)$ or Ω is a ball in R^2 and $p = r > 1, q = 1, s = 0$, that is,

$$\begin{cases} \Delta u - \mu u + \dfrac{u^p}{v} + \rho(x) = 0, & x \in \Omega, \\ \Delta v - \nu v + u^p = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

Kim [11] shown that the solution is $C^2(\Omega)\bigcap C(\overline{\Omega})$ if $p = r, q = s, q > p - 1 > 0$ and $0 < \nu < \mu$. Ghergu and Radulescu [7] proved the existence results if $r - p = s - q \geq 0$ and $p - q < 1$. Furthermore, the solution is unique if $\Omega = (0,1)$. On the other hand, the system has no positive solutions if one of the following conditions holds:

1. $s > 1$ and $2q \geq (s+1)(p+2)$.
2. $s = 1$ and $q > p + 2$.
3. $0 < s < 1$ and $q \geq p + 2$.

Later, Ghergu [5] assumed that $0 < p < 1$ and

1. if $q \leq p$ and $s \leq r$, then the system has a solution which belongs to $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$;
2. if $-1 < p - q < 0$ and $-1 < r - s < 0$, then the system has a solution which belongs to $C^2(\Omega)\bigcap C^{1,1+p-q}(\overline{\Omega}) \times C^2(\Omega)\bigcap C^{1,1+r-s}(\overline{\Omega})$.

He also obtained an interesting result for a singular and general elliptic system in [6].

In this paper, we use a new functional method to obtain the bounds for the system and then use the Leray–Schauder fixed-point theorem to prove the existence of a pair of positive solutions under suitable conditions (see the table and figure below). The new functional method is a very powerful method to obtain a priori estimates for elliptic and parabolic equations (see [1,2]).

In order to compare our results with previous ones, we list all work's $(p,\ q,\ r,\ s)$ conditions in the following table and draw Fig. 1 to show the region of solution existence for $1 < p, q < 2$:

Paper	p	q	r	s
[1]	> 1	$= 1$	$= p$	$= 0$
[11]	≥ 1	$> p-1$	$= p$	$= r$
[10]	≥ 1	$> p-1$		$= r + q - p$
This	≥ 1	$> p-1$	$> p-1$	$q - 1 < s < r + 1$ either $\geq r + \max(q - p, -1)$ or $(q-2)r/(p-1) + 1 < s < r + q - p$

It is easy to see that the regions of the previous results are only a point or a line in rs-plane.

The rest of the paper is organized as follows: In Section 2 we prove a generalized Young's inequality and an inequality related to the Laplace operator, then in Section 3 we present our main results and their proofs.