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On 2-microlocal spaces with all exponents variable

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ABSTRACT

In this paper we study various key properties for 2-microlocal Besov and Triebel–Lizorkin spaces with all exponents variable, including the lifting property, embeddings and Fourier multipliers. We also clarify and improve some statements recently published.

1. Introduction

Function spaces with variable integrability already appeared in the work of Orlicz [47], although the modern development started with the paper [35] of Kováčik and Rákosník. Corresponding PDE with non-standard growth have been studied approximately since the same time. For an overview we refer to the monographs [13,14] and the survey [26]. Apart from interesting theoretical investigations, the motivation to study such function spaces comes from applications to fluid dynamics [49], image processing [11,25,51], PDE and the calculus of variations, see for example [1,18,40].

Some ten years ago, a further step was taken by Almeida and Samko [5] and Gurka, Harjulehto and Nekvinda [24] by introducing variable exponent Bessel potential spaces $\mathcal{L}^{s,p(\cdot)}$ (with constant s). As in the classical case, these spaces coincide with the Lebesgue/Sobolev spaces for integer s. Later Xu [57] considered Besov $B^s_{p(\cdot),q}$ and Triebel–Lizorkin $F^s_{p(\cdot),q}$ spaces with variable p, but fixed q and s.

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In a different context, Leopold [36,37] considered Besov type spaces with the smoothness index determined by certain symbols of hypoelliptic pseudo-differential operators. Special choices of such symbols lead to spaces $B^{s(\cdot)}_{p,p}$ of variable smoothness. More general function spaces with variable smoothness $B^{s(\cdot)}_{p,q}$ and $F^{s(\cdot)}_{p,q}$ were explicitly studied by Besov [7], including characterizations by differences.

More recently all the above mentioned spaces were integrated into larger scales similarly with the full classical Besov and Triebel–Lizorkin scales with constant exponents. However, such extension requires all the indices to be variable. Such three-index generalization was done by Diening, Hästö and Roudenko [15] for Triebel–Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$, and by Almeida and Hästö [3] for Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$. This full extension led to immediate gains, for example with the study of traces where the integrability and smoothness indices interact, see [15,4], and it also provided an important unification. Indeed, when $s \in [0, \infty)$ and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ is bounded away from 1 and ∞ then $F^s_{p(\cdot),2} = \mathcal{L}^{s,p(\cdot)}$ [15] are Bessel potential spaces, which in turn are Sobolev spaces if s is integer [5]. On the other hand, the variable Besov scale above includes variable order Hölder–Zygmund spaces as special cases (cf. [3, Theorem 7.2]).

It happens that the smoothness parameter can be generalized in different directions. In the so-called 2-microlocal spaces $B^{\boldsymbol{w}}_{p,q}$ and $F^{\boldsymbol{w}}_{p,q}$ the smoothness is measured by a certain weight sequence $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0}$, which is rich enough in order to frame spaces with variable smoothness and spaces with generalized smoothness (see [19]). The 2-microlocal spaces already appeared in the works of Peetre [48] and Bony [8]. Later Jaffard and Meyer [28,29], and Lévy Véhel and Seuret [38] have also used such spaces in connection with the study of regularity properties of functions. Function spaces with constant integrability defined by more general microlocal weights were also studied by Andersson [6], Besov [7], Moritoh and Yamada [42] and Kempka [31].

The generalization mixing up variable integrability and 2-microlocal weights was done by Kempka [32] providing a unification for many function spaces studied so far. However, in the case of Besov spaces the fine index q was still kept fixed.

In this paper we deal with the general Besov and Triebel–Lizorkin scales $B^{w}_{p(\cdot),q(\cdot)}$ and $F^{w}_{p(\cdot),q(\cdot)}$ on $\mathbb{R}^n$ with all exponents variable. After some necessary background material (Section 2), we discuss in Section 3 the characterization in terms of Peetre maximal functions (Theorem 3.1) and, as a consequence, the independence of the spaces from the admissible system taken (Corollary 3.2). Although such statements have already been presented by Kempka and Vybíral in [33], their proofs contain some unclear points, see the details in the discussions after Theorem 3.1 and Corollary 3.3 below.

In the remaining sections we prove some key properties for both scales $B^{w}_{p(\cdot),q(\cdot)}$ and $F^{w}_{p(\cdot),q(\cdot)}$: the lifting property in Section 4; embeddings in Section 5; finally, Fourier multipliers in Section 6. For other key properties, like atomic and molecular representations and Sobolev type embeddings, we refer to our paper [2].

We notice that recently in [39] a very general framework was proposed for studying function spaces and proving similar properties for the related spaces. Although the framework suggested over there is very general in some aspects, it does not include Besov and Triebel–Lizorkin spaces with variable q. This fact is very relevant, since the mixed sequences spaces behind do not share some fundamental properties as in the constant exponent situation.

2. Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $\mathbb{Z}^n$ we denote the lattice of all points in $\mathbb{R}^n$ with integer components. If r is a real number then $r_+ := \max\{r, 0\}$. We write $B(x, r)$ for the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$. We use c as a generic positive constant, i.e. a constant whose value may change with each appearance. The expression $f \lesssim g$ means that $f \le c\, g$ for some independent constant c, and $f \approx g$ means $f \lesssim g \lesssim f$.