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Degree, instability and bifurcation of reaction–diffusion systems with obstacles near certain hyperbolas

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ABSTRACT

For a reaction–diffusion system which is subject to Turing's diffusion-driven instability and which is equipped with unilateral obstacles of various types, the nonexistence of bifurcation of stationary solutions near certain critical parameter values is proved. The result implies assertions about a related mapping degree which in turn implies for "small" obstacles the existence of a new branch of bifurcation points (spatial patterns) induced by the obstacle.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary. Consider the reaction–diffusion system

$$\begin{aligned} u_t &= d_1\Delta u + g_1(u,v) \\ v_t &= d_2\Delta v + g_2(u,v) \end{aligned} \quad \text{on } \Omega \tag{1.1}$$

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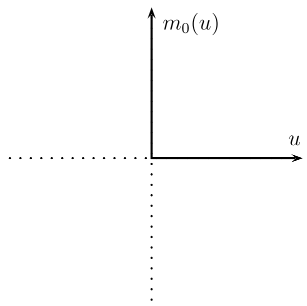

Fig. 1. Graph of m_0 (inequality type obstacle).

with boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \setminus \Gamma_1, \quad \frac{\partial u}{\partial n} \in m_1(u) \quad \text{on } \Gamma_1, \\ \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial\Omega \setminus \Gamma_2, \quad \frac{\partial v}{\partial n} \in m_2(v) \quad \text{on } \Gamma_2 \end{aligned} \tag{1.2}$$

with measurable given sets $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$.

We are interested in bifurcation of stationary solutions of (1.1), (1.2) near a constant equilibrium $(\overline{u}, \overline{v})$ which is linearly stable without diffusion terms (i.e. if $d_1 = d_2 = 0$). We assume that (1.1), (1.2) already describe the problem after a substitution which shifts this equilibrium to 0, i.e. without loss of generality, we assume that $(\overline{u}, \overline{v}) = (0, 0)$, in particular $g_1(0, 0) = g_2(0, 0) = 0$. Note that u and v then actually denote the difference to the equilibrium so that also negative values of u and v have a physical interpretation.

In case $\Gamma_1 = \Gamma_2 = \varnothing$, Turing's famous effect of "diffusion-driven instability" [13] implies that for the above problem the equilibrium $(0, 0)$ becomes instable for certain values of diffusion speeds $d_1, d_2 > 0$. Moreover, if one considers $d = (d_1, d_2) \in \mathbb{R}^2_+$ as a diffusion parameter, one typically has bifurcation of spatially nonconstant stationary solutions from the equilibrium when d crosses certain hyperbolas C_n (cf. Section 3 for details).

We are interested in the question of what changes in these observations when we impose on Γ_i obstacles described by the (in general multivalued) functions m_i. More precisely, we will assume that $m_i = m$ or $m_i(u) = -m(-u)$ where m is (or behaves in a certain weak sense near 0 qualitatively similar to) the function

$$m_0(u) := \begin{cases} [0, \infty) & \text{if } u = 0, \\ \{0\} & \text{if } u > 0, \\ \varnothing & \text{if } u < 0, \end{cases}$$

see Fig. 1. For the particular case $m_2 = m_0$, the boundary condition (1.2) becomes on Γ_2 for v the well-known Signorini boundary condition

$$v \geq 0, \qquad \frac{\partial v}{\partial n} \geq 0, \qquad \frac{\partial v}{\partial n} \cdot v = 0$$

which is the "classical" way to describe an obstacle on Γ_2: Physically, if u and v denote the difference of the concentration of two chemicals to equilibrium, this obstacle can mean that on Γ_2 the concentration v of the second chemical cannot go under the equilibrium concentration, because there is some unilateral membrane or other source which produces the corresponding chemical as soon as this would be too low. We call this an obstacle "of inequality type".

Actually, this model is physically too ideal, because the fact that $m_0(v)$ can be unbounded and empty means in the above interpretation that the source produces if necessary an unlimited amount of the chemicals and thus completely avoids the possibility that the concentration v goes under the equilibrium concentration.