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Lipschitz regularity for censored subdiffusive integro-differential equations with superfractional gradient terms

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ABSTRACT

In this paper we are interested in integro-differential elliptic and parabolic equations involving nonlocal operators with order less than one, and a gradient term whose coercivity growth makes it the leading term in the equation. We obtain Lipschitz regularity results for the associated stationary Dirichlet problem in the case when the nonlocality of the operator is confined to the domain, feature which is known in the literature as censored nonlocality. As an application of this result, we obtain strong comparison principles which allow us to prove the well-posedness of both the stationary and evolution problems, and steady/ergodic large time behavior for the associated evolution problem.

1. Introduction

In [8], the authors, in collaboration with O. Ley and S. Koike, investigate regularity properties for subsolutions of integro-differential stationary equations with a super-fractional gradient term, providing analogous results for nonlocal equations to the ones of Capuzzo-Dolcetta, Porretta and Leoni [14] for superquadratic degenerate elliptic second-order pdes (see also Barles [1]). In the present work, our aim is to obtain analogous results but for integro-differential operators of order $\sigma < 1$, still with a super-fractional gradient term, but in the (intriguing) case of *censored operators* set in bounded domains.

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In order to be more specific, we consider the model problem

$$\lambda u(x) + (-\Delta)_c^{\sigma/2} u(x) + b(x)|Du(x)|^m = f(x) \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\lambda \geq 0$ and $b, f : \bar{\Omega} \to \mathbb{R}$ are continuous functions with $b(x) > 0$ on $\bar{\Omega}$. For $\sigma \in (0,2)$, the integro-differential operator $(-\Delta)_c^{\sigma/2}$ is known in the literature as the *censored fractional Laplacian of order* σ, and is defined through the expression

$$(-\Delta)_c^{\sigma/2} u(x) = C_{N,\sigma} \text{ P.V.} \int_{x+z\in\Omega} [u(x+z) - u(x)]|z|^{-(N+\sigma)} dz, \tag{1.2}$$

where P.V. stands for the Cauchy principal value and $C_{N,\sigma} > 0$ is a well-known normalizing constant, see [13,22]. The "censored" appellative is referred to the fact that the integration on the set $\{x + z \in \Omega\}$ makes the jumps outside Ω being indeed censored.

In the scope of this paper, besides the mentioned censored nonlocality feature of the problem, the main assumptions are $0 < \sigma < 1$ (subdiffusive operator of order less than 1), and $m > \sigma$ (superfractional coercivity condition). The study of this case is motivated by two principal reasons: first, under such conditions we are able to obtain regularity properties that are (maybe surprisingly) more sophisticated than in [8]; indeed, we are not only able to obtain global Hölder continuity but also global Lipschitz regularity for bounded subsolutions of (1.1). Secondly, we obtain comparison principle and well-posedness of the Dirichlet problem both for stationary and evolution equations, namely

$$u_t + \lambda u + (-\Delta)_c^{\sigma/2} u + b(x)\ |Du|^m = f(x) \quad \text{in } \Omega \times (0,\infty). \tag{1.3}$$

Concerning Dirichlet problems for nonlocal equations, we remark that in general the *Dirichlet boundary condition* has to be imposed on the complementary of Ω. Typically, if we replace $(-\Delta)_c^{\sigma/2}$ in (1.1) or (1.3) by the fractional Laplacian $(-\Delta)^{\sigma/2}$ defined as

$$(-\Delta)^{\sigma/2} u(x) = C_{N,\sigma} \text{ P.V.} \int_{\mathbb{R}^N} [u(x+z) - u(x)]|z|^{-(N+\sigma)} dz,$$

then this requires the value of the function in the whole space to be evaluated, and the Dirichlet problem reads

$$\begin{cases} \lambda u(x) + (-\Delta)^{\sigma/2} u(x) + b(x)|Du|^m = f(x) & \text{in } \Omega, \\ u = \varphi & \text{in } \Omega^c. \end{cases} \tag{1.4}$$

On the contrary, in the censored case, since we use only the values of u in $\bar{\Omega}$, we can complement (1.1) with a *classical* boundary condition

$$u = \varphi \quad \text{on } \partial\Omega, \tag{1.5}$$

for a boundary data $\varphi \in C(\partial\Omega)$.

A third interesting question that can be handled with the development of regularity and comparison properties on the current setting is the study of the large time behavior for Cauchy–Dirichlet problem associated to (1.3). We both study the cases when the censored parabolic problem has a steady state asymptotic behavior, and the case of the *ergodic large time behavior*, situation in which we have to solve a stationary problem with state-constraint boundary condition (the *ergodic problem*).

We want to mention immediately a very important point related to our regularity and well-posedness issues (and, as a consequence, the large time behavior issues) and which justifies our choice of the parameters $0 < \sigma < 1$ and $m > \sigma$: under these conditions, we are able to solve the censored Dirichlet problem in its full generality, but this is because we use in a key way the regularity result for (1.1). On the contrary, in the case $\sigma \geq 1$ and $m > \sigma$, we are unable to prove that the censored Dirichlet problem is well-posed, even with the regularity results of [8].