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Hamilton's gradient estimates and a monotonicity formula for heat flows on metric measure spaces

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ABSTRACT

In this paper, we extend the Hamilton's gradient estimates (Hamilton 1993) and a monotonicity formula of entropy (Ni 2004) for heat flows from smooth Riemannian manifolds to (non-smooth) metric measure spaces with appropriate Riemannian curvature-dimension condition.

1. Introduction

In their fundamental work [23], Li and Yau discovered a gradient estimate for positive solutions of the heat equation

$$\partial_t u = \Delta u \tag{1.1}$$

on a smooth Riemannian manifold with Ricci curvature bounded from below. Later on, Hamilton in [16] used the similar method to establish the following gradient estimates.

Theorem 1.1 (*Hamilton [16]*). *Let (M, g) be an n-dimensional compact Riemannian manifold with Ricci curvature $R_{ij} \geqslant -Kg_{ij}$ for some $K \geqslant 0$ and $\partial M = \varnothing$. If $u(x,t)$ is a positive solution of the heat equation with $0 < u \leqslant M$. Then*

$$t|\nabla \log u|^2 \leqslant (1 + 2Kt) \cdot \log(M/u).$$

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Such type gradient estimates are called as Li–Yau–Hamilton (LYH, for short) inequalities afterwards. Recently, Kotschwar [20] extended Theorem 1.1 to complete noncompact manifolds.

The LYH inequality is of the basic tools, that has been widely used in geometric analysis. For instance, the LYH inequalities imply the classical Harnack inequalities for heat equations, by integrating the LYH type gradient estimates along space–time curves. The research of Li–Yau–Hamilton estimates for heat equation (or other geometric evolution equations) on smooth Riemannian manifolds has a long history. For an overview, the reader is referred to Chapter 4 in book [30] and [32,22,37,27,7,8], and references therein. Very recently, Bakry–Bolley–Gentil [7] have established an optimal global Li–Yau on smooth Markov semi-group under a curvature-dimension condition.

Li–Yau's gradient estimates have been extended from smooth manifolds to compact Alexandrov spaces in [28], and recently to general metric measure spaces with the *Riemannian curvature-dimension condition* $RCD^*(K,N)$, by Garofalo–Mondino [12] (for the case $\mu(X)<\infty$) and [19] (for the case $\mu(X)=\infty$). Gradient estimates for harmonic functions on metric measure spaces have been studied in [17,18]. We refer the readers to [1,3,4,6,5,11] for recent developments for Riemannian curvature dimension conditions $RCD^*(K,N)$, and [24,34,35] for curvature dimension conditions $CD(K,N)$, on metric measure spaces; see Section 2 below.

One of our main aim of this paper is to consider the LYH inequality for heat equations on non-smooth metric measure spaces (X,d,μ). Precisely, our first main result is the following:

Theorem 1.2. *Let (X,d,μ) be a proper metric measure space satisfying $RCD^*(K,\infty)$, where $K\leqslant 0$. Let $u(x,t)$ be a positive solution of the heat equation on $X\times[0,\infty)$ with initial value $u(x,0)=u_0(x)\in\cup_{1\leqslant q<\infty}L^q(X)$. Suppose that there exists a positive constant M such that $u_0(x)\leqslant M$ for almost every $x\in X$. Then*

$$t|\nabla \log u|^2\leqslant(1-2Kt)\cdot\log(M/u) \tag{1.2}$$

for almost every (x,t) in $X\times[0,\infty)$.

On the other hand, inspired by Perelman's $\mathcal{W}$-entropy, Ni [25] introduced an entropy for the heat equation on an n-dimensional smooth Riemannian manifold (M,g). For any smooth function f on M and $\tau>0$ with $\int_M(4\pi\tau)^{-n/2}e^{-f}dx=1$, the entropy is given as

$$\mathcal{W}(f,\tau):=\int_M\Big(\tau|\nabla f|^2+f-n\Big)\frac{e^{-f}}{(4\pi\tau)^{n/2}}dx.$$

Let $u(x,t)$ be a positive solution of (1.1) on a closed (i.e., compact and without boundary) manifold M with $\int_M u dx=1$. Let f be defined as $u=(4\pi\tau)^{-n/2}e^{-f}$ and $\tau=\tau(t)$ with $\frac{d\tau}{dt}=1$. In [25], Ni proved that, if M has nonnegative Ricci curvature, then the entropy $\mathcal{W}(f,\tau)$ is monotone decreasing along the heat equation (as $t\to\infty$). Such monotonicity also discussed in [10]. Wang [36] extended it to a compact manifold with Bakry–Émery curvature bounded below.

Our second main result is to extend Ni's monotonicity for heat equation to non-smooth metric measure space (X,d,μ). Precisely, we have the following result.

Theorem 1.3. *Let (X,d,μ) be a compact metric measure space satisfying $RCD^*(0,N)$ with $N\in[1,\infty]$. Let $u(x,t)$ be a positive solution of the heat equation on $X\times[0,\infty)$ with initial value $u(x,0)=u_0(x)\in L^\infty(X)$. Then we have the following:*

(i) *If $N=\infty$, by letting $f:=-\log u$, then the entropy*

$$\mathcal{W}_\infty(f,t):=\int_X|\nabla f|^2\cdot e^{-f}d\mu$$

is monotone decreasing (as $t\to\infty$).