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Uniqueness of diffusion on domains with rough boundaries

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ABSTRACT

Let Ω be a domain in $\mathbf{R}^d$ and $h(\varphi) = \sum_{k,l=1}^{d}(\partial_k\varphi, c_{kl}\partial_l\varphi)$ a quadratic form on $L_2(\Omega)$ with domain $C_c^\infty(\Omega)$ where the c_{kl} are real symmetric $L_\infty(\Omega)$-functions with $C(x) = (c_{kl}(x)) > 0$ for almost all $x \in \Omega$. Further assume there are $a, \delta > 0$ such that $a^{-1}d_\Gamma^{\,\delta}\, I \le C \le a\, d_\Gamma^{\,\delta}\, I$ for $d_\Gamma \le 1$ where d_Γ is the Euclidean distance to the boundary Γ of Ω.

We assume that Γ is Ahlfors s-regular and if s, the Hausdorff dimension of Γ, is larger or equal to $d-1$ we also assume a mild uniformity property for Ω in the neighbourhood of one $z \in \Gamma$. Then we establish that h is Markov unique, i.e. it has a unique Dirichlet form extension, if and only if $\delta \ge 1+(s-(d-1))$. The result applies to forms on Lipschitz domains or on a wide class of domains with Γ a self-similar fractal. In particular it applies to the interior or exterior of the von Koch snowflake curve in $\mathbf{R}^2$ or the complement of a uniformly disconnected set in $\mathbf{R}^d$.

1. Introduction

The theory of diffusion processes has a distinct probabilistic character and is most naturally studied on L_1-spaces. Consequently much of the analysis of such processes has relied on methods of stochastic differential equations or stochastic integration. Our aim, however, is to examine symmetric diffusion problems on domains of Euclidean space with the techniques of functional analysis and semigroup theory. In particular we focus on the characterization of uniqueness of the L_1-theory on domains with rough or fragmented boundaries. First we formulate the problem of diffusion as a problem of finding extensions of a given elliptic operator which generate semigroups with the general characteristics suited to the description of diffusion.

Let Ω be a domain in $\mathbf{R}^d$, i.e. a non-empty open connected subset, with boundary $\partial\Omega$ and $S = \{S_t\}_{t\ge0}$ a strongly continuous, positive, contraction semigroup on $L_1(\Omega)$. If the positive normalized functions in

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$L_1(\Omega)$ are viewed as probability distributions, then S has the basic properties required for description of their evolution with time. For brevity we refer to S as a diffusion semigroup. We define S to be symmetric if

$$(S_t\varphi, \psi) = (\varphi, S_t\psi) \tag{1}$$

for all $\varphi \in L_1(\Omega)$, all $\psi \in L_1(\Omega) \cap L_\infty(\Omega)$ and all $t \geq 0$. It follows that S extends by continuity from $L_1(\Omega) \cap L_\infty(\Omega)$ to a weakly* continuous semigroup on $L_\infty(\Omega)$ which we also denote by S. The extended semigroup is automatically equal to the adjoint semigroup $S^* = \{S_t^*\}_{t\geq 0}$. Then S can be defined on $L_p(\Omega)$ for each $p \in \langle 1, \infty\rangle$ by interpolation. In particular S is a self-adjoint, positive, contraction semigroup on $L_2(\Omega)$. If H is the positive, self-adjoint generator of S, it then follows from the Beurling–Deny criteria (see, for example, [24]) that the corresponding quadratic form $h(\varphi) = \|H^{1/2}\varphi\|_2^2$ with $\varphi \in D(h) = D(H^{1/2})$ is a Dirichlet form. Therefore the semigroup S is submarkovian, i.e. if $0 \leq \varphi \leq \mathbb{1}$, then $0 \leq S_t\varphi \leq \mathbb{1}$ for all $t > 0$, by the theory of Dirichlet forms [1,7].

Next define the operator H_0 on the domain $D(H_0) = C_c^\infty(\Omega)$ by

$$H_0\varphi = -\sum_{k,l=1}^{d} \partial_k \, c_{kl} \, \partial_l\varphi \tag{2}$$

where $c_{kl} = c_{lk} \in W^{1,\infty}(\Omega)$ are real and the matrix of coefficients $C(x) = (c_{kl}(x)) > 0$ for all $x \in \Omega$ in the sense of matrix order. The corresponding diffusion problem consists of classifying all extensions of H_0 in $L_1(\Omega)$ which generate symmetric diffusion semigroups. One can establish the existence of at least one such extension by quadratic form techniques. Let h_0 be the positive, quadratic form associated with H_0 on $L_2(\Omega)$, i.e.

$$h_0(\varphi) = (\varphi, H_0\varphi) = \sum_{k,l=1}^{d} (\partial_k\varphi, c_{kl} \, \partial_l\varphi) \tag{3}$$

for all $\varphi \in D(h_0) = C_c^\infty(\Omega)$. Since H_0 is a symmetric operator on $L_2(\Omega)$ the form h_0 is closable and the closure, which we denote by h_D, is automatically a Dirichlet form [1,7]. The corresponding positive, self-adjoint operator H_D, the Friedrichs' extension of H_0, generates a positive, contraction semigroup S^D on $L_2(\Omega)$ which extends to a similar semigroup on each of the L_p-spaces. The extension to $L_1(\Omega)$ automatically satisfies the symmetry relation (1). Therefore H_D generates a symmetric diffusion semigroup on $L_1(\Omega)$. The extension H_D corresponds to Dirichlet boundary conditions on $\partial\Omega$. But the same argument establishes that each Dirichlet form extension of h_0 determines the generator of a symmetric diffusion semigroup on $L_1(\Omega)$. Therefore there is a one-to-one correspondence between extensions of H_0 on $L_1(\Omega)$ which generate symmetric diffusion semigroups and Dirichlet form extensions of h_0 on $L_2(\Omega)$. The classification of extensions of H_0 which generate symmetric diffusion semigroups on $L_1(\Omega)$ is now reduced to the more amenable and transparent problem of classifying the Dirichlet form extensions of h_0 on $L_2(\Omega)$.

The Dirichlet form extensions of h_0 have a fundamental ordering property. The closure h_D is the smallest Dirichlet form extension of h_0 but there is also a largest such extension h_N. The maximal extension h_N is defined on the domain

$$D(h_N) = \{\varphi \in W^{1,2}_{\mathrm{loc}}(\Omega) : \, \Gamma(\varphi) + \varphi^2 \in L_1(\Omega)\},$$

where $\Gamma(\varphi) = \sum_{k,l=1}^d c_{kl}(\partial_k\varphi)(\partial_l\varphi)$ is the *carré du champ*, by setting

$$h_N(\varphi) = \int_\Omega dx \, \Gamma(\varphi)(x)$$

for $\varphi \in D(h_N)$. Then h_N is a Dirichlet form and the associated operator H_N is the extension of H_0 corresponding to generalized Neumann boundary conditions. But if k is a general Dirichlet form extension