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Power concavity in weakly coupled elliptic and parabolic systems

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ABSTRACT

In this paper we start the investigation of concavity properties of solutions to systems of PDE's in convex domains. In particular we prove that suitable powers of solutions to some weakly coupled elliptic and parabolic systems are concave.

1. Introduction

Consider the weakly coupled elliptic system

$$\begin{cases} -\Delta u = \lambda_1 v^\alpha & \text{in } \Omega, \\ -\Delta v = \lambda_2 u^\beta & \text{in } \Omega, \\ u > 0, \qquad v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

and the weakly coupled parabolic system

$$\begin{cases} \partial_t u - \Delta u = \lambda_1 v^\alpha & \text{in } D, \\ \partial_t v - \Delta v = \lambda_2 u^\beta & \text{in } D, \\ u > 0, \qquad v > 0 & \text{in } D, \\ u = v = 0 & \text{in } \partial D, \end{cases} \tag{1.2}$$

where $\lambda_1, \lambda_2 > 0$, $\alpha, \beta > 0$ with $\alpha\beta < 1$, Ω is a bounded convex domain in $\mathbf{R}^N$ $(N \geq 1)$ and $D := \Omega \times (0, \infty)$. In this paper we study the power concavity of solutions to systems (1.1) and (1.2).

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Concavity properties of solutions of elliptic and parabolic equations have fascinated many mathematicians and have been largely investigated. Among others, Gabriel [15] proved the convexity of the level sets of the Newtonian potential of a convex set, Makar-Limanov [37] proved that the square root of the torsion function of a planar convex set is concave, Brascamp and Lieb [6] proved the log-concavity of the first Dirichlet eigenfunction of the Laplacian as well as the preservation of log-concavity of the initial datum by the heat flow, Caffarelli and Friedman introduced in [8] the so called "constant rank" technique, as well as Alvarez, Lasry and Lions developed in [1] the technique of the convex envelope, while Korevaar established in his pioneering work [33] the concavity maximum principle for elliptic and parabolic equations and studied the concavity properties of solutions of various boundary value problems. Subsequently, Kawohl [29] and Kennington [30] extended Korevaar's approach and proved that if u is a solution of the nonlinear elliptic problem

$$-\Delta u = \lambda u^\alpha \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Omega,$$

where Ω is a bounded convex domain in $\mathbf{R}^N$, $\lambda > 0$ and $0 \le \alpha < 1$, then $v = u^{(1-\alpha)/2}$ is concave in Ω; in other words (see below), u is $\left(\frac{1-\alpha}{2}\right)$-concave. Similarly, the first and the third authors of this paper recently studied the parabolic power concavity of solutions to the parabolic boundary value problem

$$\partial_t u - \Delta u = \lambda u^\alpha \quad \text{in } D, \qquad u > 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

where Ω is a bounded convex domain in $\mathbf{R}^N$, $\lambda > 0$, $D = \Omega \times (0,\infty)$ and $0 \le \alpha < 1$, and proved in [23] that $w_\gamma(x,t) = u(x,t^{1/\gamma})^{(1-\alpha)/2}$ is concave with respect to $(x,t) \in D$ for any $\gamma \ge 1/2$; in other words (see below), u is parabolically $\left(\frac{1-\alpha}{2}\right)$-concave.

Since it would be impossible to give a complete bibliography about this subject, for further results we just refer the reader to the well known monograph by Kawohl [27] and to the papers [1–9,11,17–26,28–38], some of which are closely related to this paper and the others include recent developments in this subject.

All the above quoted results however treat scalar equations and we are not aware of any result for systems. Here instead we consider problems (1.1) and (1.2); then, as far as we know, this is the first paper dealing with concavity properties of solutions to elliptic and/or parabolic systems.

To state our results, we need first to recall the notions of power concavity and parabolic power concavity of non-negative functions. As above and throughout the paper, let Ω be a convex domain and $D = \Omega \times (0,\infty)$ and let $p > 0$. Then a positive function $u \in C(\Omega)$ is said *p-concave* if u^p is concave in Ω, while a positive function $0 < v \in C(D)$ is said *parabolically p-concave* if the function $D \ni (x,t) \mapsto v(x,t^2)^p$ is concave.

Now we are ready to state the main results of this paper.

Theorem 1.1. *Let Ω be a bounded convex domain in $\mathbf{R}^N$. Let u and v satisfy (1.1). Then u and v are p-concave and q-concave in Ω, respectively, where*

$$p = \frac{1}{2}\left(\frac{1+\beta}{1+\alpha} - \beta\right) = \frac{1-\alpha\beta}{2(1+\alpha)} \quad \textit{and} \quad q = \frac{1}{2}\left(\frac{1+\alpha}{1+\beta} - \alpha\right) = \frac{1-\alpha\beta}{2(1+\beta)}. \tag{1.3}$$

Theorem 1.2. *Let (u,v) satisfy (1.2). Then u and v are parabolically p-concave and q-concave in D, respectively, where p and q are as in (1.3).*

Notice that $0 < p, q < 1/2$ and $(p,q) \to (0,0)$ as $\alpha\beta \to 1$, while $p \to 1/2$ as $\alpha \to 0$ and $q \to 1/2$ as $\beta \to 0$ in accordance with the well known properties of the torsion function of a convex set.

To prove Theorem 1.2, we set $U = u^p$ and $V = v^q$ and construct the parabolically concave envelopes $U_{1/2}$ and $V_{1/2}$ of U and V respectively, that are, roughly speaking, the smallest parabolically concave functions greater than or equal to U and V. Then we prove that U and V coincide with $U_{1/2}$ and $V_{1/2}$. To make this, notice that $U_{1/2} \ge U$ and $V_{1/2} \ge V$ by construction, then we need to prove only the reverse inequalities. These follow from the comparison principle (see Theorem 3.1), once we prove that $(U_{1/2}, V_{1/2})$ is a subsolution