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Existence and nonexistence of positive solutions of p-Kolmogorov equations perturbed by a Hardy potential

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A B S T R A C T

In this article, we establish the phenomenon of existence and nonexistence of positive weak solutions of parabolic quasi-linear equations perturbed by a singular Hardy potential on the whole Euclidean space depending on the controllability of the given singular potential. To control the singular potential we use a weighted Hardy inequality with an optimal constant, which was recently discovered in Hauer and Rhandi (2013). Our results in this paper extend the ones in Goldstein et al. (2012) concerning a linear Kolmogorov operator significantly in several ways: firstly, by establishing existence of positive global solutions of singular parabolic equations involving nonlinear operators of p-Laplace type with a nonlinear convection term for $1 < p < \infty$, and secondly, by establishing nonexistence *locally in time* of positive weak solutions of such equations without using any growth conditions.

1. Introduction and main results

The aim of this article is to establish the phenomenon of existence and nonexistence of positive weak solutions of *p-Kolmogorov equations perturbed by a Hardy-type potential*

$$\frac{\partial u}{\partial t} - K_p u = V\ |u|^{p-2}\, u \quad \text{on } \mathbb{R}^d \times]0, T[, \tag{1.1}$$

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depending whether $\lambda \le \left(\frac{|d-p|}{p}\right)^p$ or $\lambda > \left(\frac{|d-p|}{p}\right)^p$ for $1 < p < \infty$, $d \ge 2$, and the potential $V \in L^\infty_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ satisfies

$$0 \le V(x) \le \tfrac{\lambda}{|x|^p} \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{1.2}$$

Here, we call a real-valued measurable function u on $\mathbb{R}^d \times (0,T)$ positive if $u(x,t) \ge 0$ for a.e. $x \in \mathbb{R}^d$ and a.e. $t \in (0,T)$ and the operator

$$K_p u := \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) + \rho^{-1}\,|\nabla u|^{p-2}\,\nabla u \nabla \rho \tag{1.3}$$

is the *p-Kolmogorov operator* for the particular density function

$$\rho(x) := N\, e^{-\frac{1}{p}(x^t A x)^{p/2}} \tag{1.4}$$

for every $x \in \mathbb{R}^d$, where A is a real symmetric positive definite $(d \times d)$-matrix and N some normalisation constant such as the integral $\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1$. The operator K_p was first introduced in [17] and we note that the case $A = 0$ corresponds to the density function $\rho \equiv 1$. In this case, one does not normalise and the phenomenon of existence and nonexistence of positive solutions of Eq. (1.1) on bounded and unbounded domains has been well-studied in the past (see, for instance, [15,2,19]). Thus, it is the task of this article, to investigate the case A is a real symmetric positive definite $(d \times d)$-matrix. Furthermore, we denote by $\mathrm{d}\mu$ the finite Borel-measure on $\mathbb{R}^d$ given by

$$\mathrm{d}\mu = \rho\,\mathrm{d}x,$$

for $1 \le q \le \infty$ and any open subset D of $\mathbb{R}^d$, let $L^q(D,\mu)$ and $W^{1,q}(D,\mu)$ denote the standard Lebesgue and first Sobolev space with respect to the measure $\mathrm{d}\mu$ and $W^{1,q}_0(D,\mu)$ the closure of $C^\infty_c(D)$ in $W^{1,q}(D,\mu)$. Under these assumptions, the second and third authors of this article established in [21] the following Hardy inequality with a remainder term.

Lemma 1.1 *([21]). Let $d \ge 2$, $1 < p < \infty$ and A be a real symmetric positive definite $(d \times d)$-matrix. Then*

$$\left(\tfrac{|d-p|}{p}\right)^p \int_{\mathbb{R}^d} \tfrac{|u|^p}{|x|^p}\,\mathrm{d}\mu \le \int_{\mathbb{R}^d} |\nabla u|^p\,\mathrm{d}\mu + \left(\tfrac{|d-p|}{p}\right)^{p-1} \operatorname{sign}(d-p) \int_{\mathbb{R}^d} |u|^p\,\frac{(x^t A x)^{p/2}}{|x|^p}\,\mathrm{d}\mu \tag{1.5}$$

for all $u \in W^{1,p}(\mathbb{R}^d,\mu)$ with optimal constant $\left(\frac{|d-p|}{p}\right)^p$.

In contrast to the case $A \equiv 0$ (cf., for instance, [15] or [26] and the references therein), our weighted Hardy inequality (1.5) admits the remainder term

$$\left(\tfrac{|d-p|}{p}\right)^{p-1} \operatorname{sign}(d-p) \int_{\mathbb{R}^d} |u|^p\,\frac{(x^t A x)^{p/2}}{|x|^p}\,\mathrm{d}\mu. \tag{1.6}$$

This term has, in fact, a great impact on the existence of weak solutions of Eq. (1.1) in the degenerate case $2 < p < d$, while for establishing nonexistence of positive solutions this term does not cause any problems. It is somehow surprising that in the case $p > d$, the remainder term (1.6) becomes negative and so provides further estimates in $L^p(\mathbb{R}^d,\mu)$. We note that one does not find much in the literature about Hardy type inequalities in the case $p > d \ge 2$.

In this article, we make use of the following notion of *weak solutions*, which seems to be natural for parabolic equations of p-Laplace type with singular potentials (cf. [10,8,9] or [18] for $p = 2$ and [19] by J. Goldstein and Kombe).

Definition 1.2. Let $V \in L^\infty_{\text{loc}}(\mathbb{R}^d \setminus \{0\},\mu)$ be positive. If $p \ne 2$, then for given $u_0 \in L^2_{\text{loc}}(\mathbb{R}^d,\mu)$ we call u a *weak solution* of Eq. (1.1) with initial value $u(0) = u_0$ provided

$$u \in C([0,T); L^2_{\text{loc}}(\mathbb{R}^d \setminus \{0\},\mu)) \cap L^p(0,T; W^{1,p}_{\text{loc}}(\mathbb{R}^d \setminus \{0\},\mu)),$$