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Non-uniqueness for second order elliptic operators

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ABSTRACT

We show an abstract method to construct different realizations of abstract operators generating a strongly continuous semigroup and apply it to several second order elliptic operators with singular coefficients.

1. Introduction

In this paper we show how to construct infinitely many realizations of the Schrödinger operator with inverse square potential $S = \Delta - \frac{b}{|x|^2}$ generating an analytic and positive semigroup, when smooth functions with compact support away from the origin do not constitute a core. We work in $L^p(\mathbb{R}^N)$ and define the maximal operator

$$\begin{cases} S_{p,\max} := Su \quad \text{as a distribution in } \mathbb{R}^N \setminus \{0\}, \\ D(S_{p,\max}) := \{u \in L^p(\mathbb{R}^N);\ Su \in L^p(\mathbb{R}^N)\} \end{cases} \tag{1.1}$$

and the minimal operator

$$\begin{cases} S_{p,\min} u := Su, \\ D(S_{p,\min}) := \text{the closure of } S \text{ defined in } C_0^\infty(\mathbb{R}^N \setminus \{0\}). \end{cases} \tag{1.2}$$

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The L^2-analysis of the operator is the following. By Hardy's inequality, $S_{2,\min}$ is form-bounded if and only if $b+\left(\frac{N-2}{2}\right)^2 \geq 0$ and, by Rellich inequality, $S_{2,\min}$ is self-adjoint if and only if $b+\frac{N(N-4)}{4} = b+\left(\frac{N-2}{2}\right)^2-1 \geq 0$ (see e.g., Reed–Simon [14, Chapter X]). When $b+\left(\frac{N-2}{2}\right)^2 < 1$, therefore, there are infinitely many self-adjoint extensions S such that $S_{2,\min} \subsetneq S \subsetneq S^*_{2,\min} = S_{2,\max}$.

We also recall that the condition $b+(\frac{N-2}{2})^2 \geq 0$ is necessary and sufficient for the existence of positive distributional solutions of the corresponding parabolic problem, due to a famous result in [3]. However, when $b+(\frac{N-2}{2})^2 < 0$, one can construct non-self-adjoint realizations of S which generate an analytic semigroup in $L^2(\mathbb{R}^N)$, see [9].

The L^p-analysis, $1 < p < \infty$, is the following. Under the assumption $b+\left(\frac{N-2}{2}\right)^2 \geq 0$ and setting

$$s_1 = \frac{N-2}{2} - \sqrt{b+\left(\frac{N-2}{2}\right)^2}, \qquad s_2 = \frac{N-2}{2} + \sqrt{b+\left(\frac{N-2}{2}\right)^2}, \tag{1.3}$$

it is proved in [8, Section 7] and [10] that there exists a realization $S_{p,\min} \subset S_{p,\text{int}} \subset S_{p,\max}$ generating a semigroup in $L^p(\mathbb{R}^N)$ if and only if $\frac{N}{p} \in (s_1, s_2+2)$; the generated semigroup is analytic and positive. When $p=2$, $S_{2,\text{int}}$ coincides with the Friedrichs extension of $S_{2,\min}$. Moreover, $S_{p,\text{int}} = S_{p,\max}$ if and only if $\frac{N}{p} \in (s_1, s_2]$ and $S_{p,\text{int}} = S_{p,\min}$ if and only if $\frac{N}{p} \in [s_1+2, s_2+2)$. Therefore $S_{p,\text{int}}$ is the unique realization of S between $S_{p,\min}$ and $S_{p,\max}$ which generates a semigroup, when these two intervals overlap, that is when $s_1+2 \leq s_2$, since it coincides either with $S_{p,\min}$ or with $S_{p,\max}$ (and with both when $\frac{N}{p} \in [s_1+2, s_2]$). The problem of uniqueness arises only when $s_2 < s_1+2$ and N/p is in between, that is when

$$b+\left(\frac{N-2}{2}\right)^2 \in [0,1) \quad \text{and} \quad \frac{N}{p} \in (s_2, s_1+2).$$

In this paper we prove that, under the above assumption, there exist infinitely many realizations between $S_{p,\min}$ and $S_{p,\max}$ which generate positive and analytic semigroups of angle $\frac{\pi}{2}$ in $L^p(\mathbb{R}^N)$. This result seems to be new also for $p=2$, concerning positivity. Note that the situation does depend on p. When $b+\left(\frac{N-2}{2}\right)^2 \in [0,1)$, the semigroup generated by $S_{p,\text{int}}$ acts in all L^p where $\frac{N}{p} \in (s_1, s_2+2)$ but other analytic and positive semigroups appear only for those p such that $\frac{N}{p} \in (s_2, s_1+2)$ and there is uniqueness (in the above sense) only for certain p.

In order to get the above result, we describe a general method to construct different realizations of abstract operators generating strongly continuous semigroups. We consider densely defined operators A_0, $A_{\min}$ and $A_{\max}$ in a Banach space X satisfying $A_{\min} \subsetneq A_0 \subsetneq A_{\max}$ and $\rho(A_0) \neq \varnothing$ and we show how to construct infinitely many realizations between $A_{\min}$ and $A_{\max}$ different from A_0, with non-empty resolvent set. In some cases we show that these realizations generate positive or analytic semigroups.

We point out that this uniqueness/non-uniqueness problem is different from the classical core problem considered in [5], even when we do not consider positivity. If A_0 generates a semigroup $(T(t))_{t\geq 0}$, then $A_{\min}$ is a core for A_0 if and only if $(T(t))_{t\geq 0}$ is the only semigroup whose generator extends $A_{\min}$, see [2, Theorem 1.33]. However, when $D(A_{\min})$ is not a core, an extension generating a semigroup need not be a restriction of $A_{\max}$. The proof of [2, Theorem 1.33] shows that extensions can be constructed by preserving the domain of A_0 but changing the operator on $D(A_0) \setminus D(A_{\min})$. This is not our point of view since, looking for extensions always contained in the maximal operator, we keep fixed the expression of the differential operator on the larger space of distributions. In the above situation, moreover, we have uniqueness even when $S_{p,\max}$ is a generator but $D(S_{p,\min})$ is not a core.

Since our approach is quite flexible, in Section 4 we apply it to a more general differential operator whose coefficients can be singular at 0 and ∞.