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The Matzoh Ball Soup Problem: A complete characterization ☆

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ABSTRACT

We characterize all the solutions of the heat equation that have their (spatial) equipotential surfaces which do not vary with the time. Such solutions are either isoparametric or split in space–time. The result gives a final answer to a problem raised by M.S. Klamkin, extended by G. Alessandrini, and that was named the *Matzoh Ball Soup Problem* by L. Zalcman. Similar results can also be drawn for a class of quasi-linear parabolic partial differential equations with coefficients which are homogeneous functions of the gradient variable. This class contains the (isotropic or anisotropic) evolution p-Laplace and normalized p-Laplace equations.

1. Introduction

In this paper we settle a question raised by M.S. Klamkin in [5] and extended in [2].

Klamkin's conjecture (1964). Consider the heat conduction problem for a solid Ω,

$$u_t = \Delta u \quad \text{in } \Omega \times (0, \infty).$$

Initially, $u = 0$. On the boundary $u = 1$. The solution to the problem is well-known for a sphere and, as to be expected, it is radially symmetric. Consequently, the equipotential surfaces do not vary with the time (the temperature on them, of course, varies). It is conjectured for the boundary value problem above, that the sphere is the only bounded solid having the property of invariant equipotential surfaces. If we allow unbounded solids, then another solution is the infinite right circular cylinder which corresponds to the spherical solution in two-dimensions.

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L. Zalcman [19] included this problem in a list of questions about the ball and named it the *Matzoh Ball Soup Problem*. For the case of a bounded solid (by solid we mean, according to [1,19], a connected open set that coincides with the interior of its closure and whose complement is connected), the conjecture was given a positive answer by G. Alessandrini [1]: *the ball is the only bounded solid having the property of invariant equipotential surfaces.*

In [10], the second author of this paper and S. Sakaguchi showed that to obtain the spherical symmetry of the solid in Klamkin's setting, it is enough to require that the solution has *only one* invariant equipotential surface (provided this surface is a C^1-regular boundary of domain). In a subsequent series of papers, the same authors extended their result in several directions: spherical symmetry also holds for certain evolution nonlinear equations [11,13,15,16]; a hyperplane can be characterized as an invariant equipotential surface in the case of an unbounded solid that satisfies suitable sufficient conditions [12,14]; for a certain Cauchy problem, a helicoid is a possible invariant equipotential surface [9]; spheres, infinite cylinders and planes are characterized as (single) invariant equipotential surfaces in $\mathbb{R}^3$ [8]; similar symmetry results can also be proven in the sphere and the hyperbolic space [13].

In [2], G. Alessandrini re-considered Klamkin's problem for a bounded domain in the case in which u initially equals any function $u_0 \in L^2(\Omega)$ and is zero on $\partial\Omega$ for all times. He discovered that either u_0 is a Dirichlet eigenfunction or Ω is a ball. A comparable result was obtained by S. Sakaguchi [17] when a homogeneous Neumann condition is in force on $\partial\Omega$.

The aim of this paper is to show that Klamkin's property of having invariant equipotential surfaces characterizes a solution of the heat equation without assuming *any whatsoever* initial or boundary condition. This is the content of our main result.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a domain and let u be a solution of the heat equation:*

$$u_t = \Delta u \quad \text{in } \Omega \times (0, \infty). \tag{1.1}$$

Assume that there exists a $\tau > 0$ such that, for every $t > \tau$, $u(\cdot, t)$ is constant on the level surfaces of $u(\cdot, \tau)$ and $Du(\cdot, \tau) \neq 0$ in Ω.

Then one of the following occurrences holds:

(i) *the function $\varphi = u(\cdot, \tau)$ (and hence u) is isoparametric, that is there exist two real-valued functions f and g such that φ is a solution of the following system of equations:*

$$|D\varphi|^2 = f(\varphi) \quad \text{and} \quad \Delta\varphi = g(\varphi) \quad \text{in } \Omega;$$

(ii) *there exist two real numbers λ, μ such that*

$$u(x, t) = e^{-\lambda t}\phi_\lambda(x) + \mu, \quad (x, t) \in \Omega \times [\tau, \infty),$$

where

$$\Delta\phi_\lambda + \lambda\,\phi_\lambda = 0 \quad \text{in } \Omega;$$

(iii) *there exists a real number γ such that*

$$u(x, t) = \gamma\, t + w(x), \quad (x, t) \in \Omega \times [\tau, \infty),$$

where

$$\Delta w = \gamma \quad \text{in } \Omega.$$