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On the Cauchy problem for non-local Ornstein–Uhlenbeck operators

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ABSTRACT

We study the Cauchy problem involving non-local Ornstein–Uhlenbeck operators in finite and infinite dimensions. We prove classical solvability without requiring that the Lévy measure corresponding to the large jumps part has a first finite moment. Moreover, we determine a core of regular functions which is invariant for the associated transition Markov semigroup. Such a core allows to characterize the marginal laws of the Ornstein–Uhlenbeck stochastic process as unique solutions to Fokker–Planck–Kolmogorov equations for measures.

1. Introduction and notation

In this paper we investigate solvability of the Cauchy problem involving non-local Ornstein–Uhlenbeck operators both in finite and infinite dimensions. We also determine a core of regular functions which is invariant for the transition Ornstein–Uhlenbeck semigroup. Differently with respect to recent papers (see [1, Section 5], [19, Section 4.1] and [37, Section 2]) to study the core problem we do not require that the associated Lévy measure ν corresponding to the large jumps part has a first finite moment (see (8)).

Let us first introduce the Ornstein–Uhlenbeck operator $\mathcal{L}_0$ in $\mathbb{R}^d$ and its associated stochastic process. The operator $\mathcal{L}_0$ is defined as

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$$\mathcal{L}_0 f(x) = \frac{1}{2}\sum_{j,k=1}^{d} Q_{jk}\,\partial^2_{x_j x_k} f(x) + \sum_{j=1}^{d} a_j \partial_{x_j} f(x) + \sum_{j,k=1}^{d} A_{jk}\, x_k \partial_{x_j} f(x)$$
$$+ \int_{\mathbb{R}^d} \Big(f(x+y) - f(x) - \mathbb{1}_{\{|y|\le 1\}}\,(y) \sum_{j=1}^{d} y_j\,\partial_{x_j} f(x) \Big)\nu(\mathrm{d}y), \quad x \in \mathbb{R}^d, \tag{1}$$

where $\mathbb{1}_{\{|y|\le 1\}}$ is the indicator function of the closed ball with center 0 and radius 1, $Q = (Q_{ij})$ and $A = (A_{ij})$ are given $d \times d$ real matrices (Q being symmetric and non-negative definite). Moreover $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and ν is a Lévy jump measure, i.e., ν is a σ-finite Borel measure on $\mathbb{R}^d$ such that

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \wedge |y|^2)\,\nu(dy) < \infty \tag{2}$$

($a \wedge b$ indicates the minimum between a and $b \in \mathbb{R}$). The function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $C^2_b(\mathbb{R}^d)$ (i.e., f is bounded and continuous together with its first and second partial derivatives) and the integral in (1) is well defined thanks to the Taylor formula. The associated Ornstein–Uhlenbeck process (OU process) solves the following SDE driven by a Lévy process Z:

$$\begin{cases} \mathrm{d}X_t = AX_t\mathrm{d}t + \mathrm{d}Z_t, & t \ge 0 \\ X_0 = x, & x \in \mathbb{R}^d \end{cases} \tag{3}$$

(see, for instance, [33,32,24]). The matrix A is the same as in (1) and $Z = (Z_t)_{t\ge 0} = (Z_t)$ is a d-dimensional Lévy process uniquely determined in law by the previous Q, a and ν (cf. [31, Section 9]). Ornstein–Uhlenbeck processes with jumps $X = (X_t)$ have many applications to Mathematical Finance and Physics (see for instance, [3,10,16,18]); moreover in some interesting cases the Lévy measure ν corresponding to the large jumps part has not first finite moment (see [16,18] when $\alpha \in (0,1]$ and the references therein). For example, in [16] an Ornstein–Uhlenbeck process driven by a Cauchy process is considered in the context of anomalous diffusions.

The corresponding transition Markov semigroup $(P_t) = (P_t)_{t\ge 0}$ is called the Ornstein–Uhlenbeck semigroup (or Mehler semigroup):

$$P_t f(x) = \mathbb{E}[f(X^x_t)], \quad t \ge 0, \tag{4}$$

for any $f : \mathbb{R}^d \to \mathbb{R}$ which is Borel and bounded (see also (14)). In Section 3.1 we prove well-posedness of the Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \mathcal{L}_0 u(t,x) \\ u(0,x) = f(x), \quad x \in \mathbb{R}^d,\ f \in C^2_b(\mathbb{R}^d), \end{cases} \tag{5}$$

where $\mathcal{L}_0 u(t,x) = (\mathcal{L}_0 u(t,\cdot))(x)$ (see Theorem 3.3). We show that there exists a unique bounded classical solution given by $u(t,x) = P_t f(x)$, $t \ge 0$, $x \in \mathbb{R}^d$. Our result is not covered by regularity results on singular pseudodifferential operators (cf. [20,25] and Remark 3.4). Moreover, it cannot be deduced by standard arguments of semigroup theory using known results for (P_t) (see, in particular, [33,24,29] and Remark 3.7). To prove solvability of (5) we first establish the crucial formula

$$P_t f(x) = f(x) + \int_0^t \mathcal{L}_0(P_s f)(x)\mathrm{d}s, \quad t \ge 0, x \in \mathbb{R}^d,\ f \in C^2_b(\mathbb{R}^d) \tag{6}$$

(see Theorem 3.1). Note that in [33, Theorem 3.1] it is proved that

$$P_t f(x) = f(x) + \int_0^t P_s(\mathcal{L}_0 f)(x)\mathrm{d}s, \quad t \ge 0, x \in \mathbb{R}^d,\ f \in C^2_K(\mathbb{R}^d) \tag{7}$$