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# Exponential decay for a nonlinear model for electrical conduction in biological tissues

M. Amar, D. Andreucci*, R. Gianni

*Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Università di Roma "La Sapienza", Via A. Scarpa 16, 00161 Roma, Italy*

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ABSTRACT

We study the asymptotic convergence to a periodic steady state of the solution of a nonlinear system of equations with periodic boundary data modeling electrical conduction in biological tissues, both in the microscopic and in the homogenized version. Such model keeps into account the resistive behavior of the intracellular and extracellular domains and also the capacitive/resistive behavior of the lipidic cellular membrane.

The rate of convergence is analyzed and the systems of equations satisfied by the asymptotic limits are exhibited, when the resistive behavior of the membrane is described by a strictly monotone and coercive nonlinear function.

The special case of homogeneous boundary conditions is also investigated, where the coercivity assumption can be relaxed.



## 1. Introduction

Composite materials have widespread applications in science and technology and, for this reason, have been extensively studied especially using homogenization techniques (we quote, among others, [15,17,19,31,32,35,36]). In this framework the authors, and co-workers, have investigated a problem arising in electric conduction in biological tissues with the purpose of obtaining some useful results for applications in electrical tomography (see [4–12]).

We deal with the physical problem of electric currents crossing a living tissue when an electrical potential is applied at the boundary (see [16,18,21,25,29]). Here the living tissue is regarded as a composite periodic domain made of extracellular and intracellular phases both assumed to be conductive, possibly with different conductivities, separated by a lipidic membrane experimentally found to exhibit both conductive (due to ionic channels in the membrane) and capacitive behavior. In this regard the large number of cells contained

* Corresponding author.
*E-mail addresses:* micol.amar@sbai.uniroma1.it (M. Amar), daniele.andreucci@sbai.uniroma1.it (D. Andreucci), roberto.gianni@sbai.uniroma1.it (R. Gianni).

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in the biological sample allows us and even imposes to use a homogenization technique. Such technique yields the system of partial differential equations satisfied by the macroscopic electric potential $u$, which is the limit of the electric potential $u_\varepsilon$ as the characteristic length of the cell $\varepsilon$ tends to zero.

Clearly if we want the capacitive and the conductive behavior of the membranes to be maintained when $\varepsilon \to 0$ we must properly rescale the capacity and the conductivity of such membranes with respect to $\varepsilon$. In [8,12] the authors have shown that, essentially, only three scalings are physically reasonable. One of these scalings seems to be the most suitable to describe the behavior of the membranes for currents in the radiofrequency range (which is the standard frequency used in electrical tomography). In this model the magnetic field is neglected (as suggested by experimental evidence) and the potential $u_\varepsilon$ is assumed to satisfy an elliptic equation both in the intracellular and in the extracellular domain while on the membranes it satisfies the equation

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] + f\left(\frac{[u_\varepsilon]}{\varepsilon}\right) = \sigma^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon ,$$

where $[u_\varepsilon]$ denotes the jump of the potential across the membranes and $\sigma^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon$ is the current crossing the membranes. The interface condition above was rigorously obtained in [8] by means of a concentrated capacity technique, whence the onset of the scaling specific to this model.

From a mathematical point of view the cases of linear or nonlinear $f$ are markedly different. Homogenization limits have been rigorously found in both cases. The linear case has been considered in [4,6,11], via asymptotic expansion in $\varepsilon$. It has been shown that the limit potential $u$ satisfies an elliptic equation with memory for which an existence and uniqueness theorem has been proved in [5].

In the nonlinear case the approach is much more complicated and relies on the two-scale convergence technique [2,3,20,28,34]. In this case a memory effect is still present in the limiting problem, which however does not take the form of a single partial differential equation satisfied by $u$ (see [12]). Indeed, the problem rather contains two unknowns $u$ and $u_1$; the latter accounts for the microscopic properties of the material and depends both on the macroscopic variable $x$ and on the microscopic variable $y$. Formally this limiting problem keeps the abstract parabolic structure which is characteristic of the microscopic scheme described above and already remarked in [9].

For a physical and biological motivation of the interest in nonlinear models of the type considered in this paper see for instance [14,30,33,1].

Going back to the technical applications of bioimpedance tomography it must be noted that usually the applied boundary potential is time harmonic, allowing for the empirical assumption that the resulting potential inside the biological material is time harmonic too. Under this assumption the behavior of the biological tissue is modeled by means of complex elliptic equations, one for every harmonic frequency. The correctness of this model has been proved by the authors in the linear case in [9–11], investigating the time limit, as $t \to +\infty$, of the solution $u$ of the homogenized problem. There it was proved that the equations currently used in electrical tomography can be rigorously obtained by means of an asymptotic limit with respect to $t$ when time periodic boundary data are assigned. Here the asymptotic decay is exponential. As a new input, those papers revealed the relation expressing the complex admittivity of the limiting equation as an explicit function of the frequency of the boundary data and of the physical properties of the tissue.

It is remarkable that an elliptic equation with memory in general does not exhibit asymptotic stability even if the memory kernel decreases to zero exponentially when $t \to +\infty$ (see [24]). For this reason, in the papers quoted above the result is obtained proving an asymptotic exponential convergence in $t$ for the problem of level $\varepsilon$ (i.e., before homogenization) and observing that such a convergence is stable with respect to $\varepsilon$, so that it holds true also for the limiting potential. For an alternative approach to asymptotic stability in problems with memory, relying on some extra-assumptions on the structure of the kernel see, for instance, [22,23]. About asymptotic decay in diffusive or electromagnetic systems see also [26,27] and the references therein.