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Surgery of the Faber–Krahn inequality and applications to heat kernel bounds

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1. Introduction

Let M be a non-compact connected Riemannian manifold and let μ be the Riemannian measure on M. For each non-empty open subset $\Omega \subset M$, denote by $\lambda(\Omega)$ the first eigenvalue of the Dirichlet problem in Ω for the Laplace–Beltrami operator Δ. The *Faber–Krahn inequality* is a lower bound on $\lambda(\Omega)$ in terms of the volume $\mu(\Omega)$ as follows:

$$\lambda(\Omega) \geq \Lambda(\mu(\Omega)), \tag{1.1}$$

where Λ is a non-negative function on $(0, +\infty)$. The function Λ is called the *Faber–Krahn function* of an open set $U \subset M$ if (1.1) holds for all $\Omega \subset U$. Since $\lambda(\Omega)$ decreases on expansion of Ω, we will always assume that the Faber–Krahn function is monotone decreasing.

Recall that the classical Faber–Krahn theorem states that, for any open set $\Omega \subset \mathbb{R}^N$,

$$\lambda(\Omega) \geq \lambda(B),$$

where B is the Euclidean ball with volume $\mu(B) = \mu(\Omega)$. It is easy to see that $\lambda(B) = c_N \mu(B)^{-2/N}$. Hence, according to the definition given above, $\mathbb{R}^N$ has the Faber–Krahn function

$$\Lambda(v) = c_N v^{-2/N}.$$

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This paper describes how Faber–Krahn inequalities and Faber–Krahn functions behave under removal of a compact set with smooth boundary (Section 2.2, Proposition 2.1 and Theorem 2.4) and under gluing of several non-compact manifolds (Section 3, Theorem 3.3). This is somewhat a technical goal but these results should prove useful in various situations. In particular, they extend (in a sense) those of [2] where a Sobolev inequality for the exterior of certain compact domains in $\mathbb{R}^N$ was proved.

One specific application of these cutting and gluing results is presented in detail in Section 4. It concerns with the problem of estimating of the heat kernel on a manifold with ends. To describe more precisely this application, let us assume that M is geodesically complete and let $K \subset M$ be a compact set with smooth boundary such that $M \setminus K$ has k connected components $E_1, \ldots, E_k$. The sets E_i are called the *ends* of M with respect to K.

Furthermore, in many cases each end E_i can be considered as the exterior of a compact set with smooth boundary in another complete manifold M_i. In this case we say that M is a *connected sum* of $M_1, \ldots, M_k$ and write

$$M = \bigsqcup_{i=1}^{k} M_i$$

(see Section 3.1 for a careful definition).

Now, suppose that each M_i is a non-compact complete manifold for which we have a good heat kernel upper bound. What information can we obtain for the heat kernel on the connected sum M?

The study of the relationships between heat kernel bounds and functional inequalities (such as Faber–Krahn inequalities and others) has been an active area of research during the past decades (see, e.g., [4,21,8,11]). In view of the previous experience it is natural to attack the above question about heat kernel bounds on connected sums of manifolds by using the Faber–Krahn inequalities, which is done in this paper.

We obtain fairly satisfactory heat kernel bounds that are easy to apply in some cases. For example, let us consider the special case when each end E_i is the exterior of a compact with smooth boundary in a non-compact complete manifold M_i with non-negative Ricci curvature. Let $V_i(x, r)$ be the volume of the geodesic ball in M_i of radius r and center $x \in M_i$. For any $r > 0$, set

$$V_{\min}(r) = \min_{1 \le i \le k} V_i(o_i, r),$$

where $o_i \in \partial E_i$ is a fixed reference point. In this situation we prove that, for all $t > 0$,

$$\sup_{x,y \in K} p(t, x, y) \le \frac{C}{V_{\min}(\sqrt{t})}. \tag{1.2}$$

(see Theorem 4.5). The estimate (1.2) is used in our paper [13] as a key ingredient for obtaining two-sided estimates of $p(t, x, y)$ for the full range $x, y \in M$ and $t > 0$ in the above setting. In particular, it follows from [13] that (1.2) is sharp, that is, has a matching lower bound, provided each manifold M_i is non-parabolic.

We denote by the letters c, C, c', C' etc. positive constants whose values can change at each occurrence.

2. Cutting Faber–Krahn inequalities

In this section we show that the Faber–Krahn inequality is roughly preserved under the removal of a compact set with smooth boundary.

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