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Propagation and extinction of fronts for semilinear parabolic equations on Riemannian manifolds

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ABSTRACT

We study the Cauchy problem for the semilinear heat equation on Riemannian manifolds. Propagation and extinction of solutions are addressed, supposing that the nonlinear forcing term is either of KPP type or of bistable type. In particular, we highlight the influence both of sectional curvatures and of Ricci curvature on such phenomena.

1. Introduction

We are concerned with the Cauchy problem for semilinear parabolic equations of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } M \times (0, \infty) \\ u = u_0 & \text{in } M \times \{0\}, \end{cases} \tag{1.1}$$

where M is an n-dimensional complete, noncompact Riemannian manifold, and Δ is the Laplace–Beltrami operator on M. Concerning the nonlinear forcing term f, we always assume:

$$f \in C^1([0,1]), \qquad f(0) = f(1) = 0. \tag{H_0}$$

More precisely, we mainly consider f of two types:

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- the "KPP" type, i.e.

$$f'(0) > 0, \qquad f(u) > 0 \quad \text{for all } u \in (0,1) \tag{H_1}$$

- the "bistable" type, i.e.

$$\begin{cases} \text{(i) there exists } a \in (0,1) \text{ such that } f(u) < 0 \text{ for all } u \in (0,a),\ f(u) > 0 \text{ for all } u \in (a,1); \\ \text{(ii) } f'(0) < 0, \qquad \int_0^1 f(u)du > 0. \end{cases} \tag{H_2}$$

On the other hand, regarding the initial datum u_0, the following assumption will always be made in the sequel:

$$u_0 \quad \text{continuous in } M, \qquad 0 \le u_0(x) \le 1 \quad \text{for all } x \in M. \tag{H_3}$$

In addition, we always suppose that the Ricci curvature is bounded from below by $-C(1+\text{dist}(x,o)^2)$ for some positive constant C and for some fixed point $o \in M$. This condition implies that M is *stochastically complete*; moreover, the comparison principle for problem (1.1) holds (see [8, Corollary 15.2]), see also [18, Proposition 3.5, Remark 3.6]. Therefore, by assumptions (H_0), (H_3) and the comparison principle, every solution u of problem (1.1) satisfies the inequality

$$0 \le u(x,t) \le 1 \quad \text{for all } (x,t) \in M \times (0,\infty). \tag{1.2}$$

We always deal with classical solutions of problem (1.1). In this respect, it is easily seen that under assumptions (H_0), (H_3) a unique solution of problem (1.1) exists. In fact, the existence follows by the a priori estimate (1.2) and standard compactness arguments (see *e.g.* [17]), while the uniqueness by the comparison principles mentioned above.

The counterpart of problem (1.1) in the Euclidean space $\mathbb{R}^n$, that is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = u_0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{1.3}$$

has been subject to detailed investigations in the literature (see, *e.g.*, [1–3,5,7,10,13,15]); moreover, it has a great interest for applications, especially in Mathematical Biology (see [3]). Clearly, if one considers such diffusion phenomena on possibly curved space, then they can be described by problem (1.1). Consequently, it is natural to investigate the influence of the geometry of the underlying space on qualitative properties of solutions. In this direction, recently in [14] it has been addressed problem (1.1) in the special case that M is the n-dimensional hyperbolic space $\mathbb{H}^n$, that is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \mathbb{H}^n \times (0,\infty) \\ u = u_0 & \text{in } \mathbb{H}^n \times \{0\}. \end{cases} \tag{1.4}$$

Let us now recall some results from [3,14]. In [3] it is shown that if the forcing term f is of KPP type, then *propagation* always occurs, namely

$$\lim_{t\to\infty} u(x,t) = 1 \quad \text{uniformly on compact subsets of } \mathbb{R}^n, \tag{1.5}$$

for every solution u of problem (1.3) with $u_0 \not\equiv 0$. On the other hand, if f is of bistable type, there is a "threshold effect". In fact, there is *extinction*, that is

$$\lim_{t\to\infty} u(x,t) = 0 \quad \text{uniformly on in } \mathbb{R}^n, \tag{1.6}$$