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The Cauchy problem and blow-up phenomena of a new integrable two-component Camassa–Holm system

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This paper considers the Cauchy problem and blow-up phenomena of a new integrable two-component Camassa–Holm system, which is a natural extension of the Fokas–Olver–Rosenau–Qiao equation. Firstly, the local well-posedness of the system in the critical Besov space $B^{\frac{1}{2}}_{2,1}(\mathbb{R}) \times B^{\frac{1}{2}}_{2,1}(\mathbb{R})$ is investigated, and it is shown that the data-to-solution mapping is Hölder continuous. Then, a blow-up criteria for the Cauchy problem in the critical Besov space is derived. Moreover, with conditions on the initial data, a new blow-up criteria is obtained by virtue of the blow-up criteria at hand and the conservative property of m and n along the characteristics. Finally, a global existence result for the strong solution is established.

1. Introduction

In this paper, we consider the Cauchy problem and the blow-up phenomena of a new two-component Camassa–Holm system with cubic nonlinearities,

$$\begin{cases} m_t + ((uv - u_x v_x) - (uv_x - u_x v))m_x = -((u_x n + v_x m) + (un - vm))m, \quad x \in \mathbb{R},\ t \geq 0, \\ n_t + ((uv - u_x v_x) - (uv_x - u_x v))n_x = -((u_x n + v_x m) + (un - vm))n, \quad x \in \mathbb{R},\ t \geq 0, \\ m(x,0) = m_0(x), \quad x \in \mathbb{R}, \\ n(x,0) = n_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where $m = u - u_{xx}$ and $n = v - v_{xx}$. The system (1.1) was recently proposed by Song, Qu and Qiao [39], in which the authors shown that system (1.1) possesses the Lax-pair

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = \begin{pmatrix}\frac{1}{2} & \lambda m\\ \lambda n & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{1.2}$$

and

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = \begin{pmatrix}-\frac{1}{4}\lambda^{-2} + \frac{1}{2}Q & \frac{1}{2}\lambda^{-1}(u+u_x) + \lambda m Q\\ \frac{1}{2}\lambda^{-1}(v-v_x) + \lambda n Q & -\frac{1}{4}\lambda^{-2} - \frac{1}{2}Q\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{1.3}$$

where $Q = u_x v_x - uv + uv_x - u_x v$, this fact implies that the system (1.1) is integrable. Moreover, it was proved that the system (1.1) is also geometrically integrable, in other words, the system describes the pseudospherical surfaces. As a consequence, its infinite number of conservation laws can be constructed through some recursion relations. The explicit solutions of (1.1) such as the cuspons and W/M-shape solitons are also obtained [39]. Very recently, Yan et al. [42] studied the one peaked solitons and two peakon solitons by means of an explicit formula. Meanwhile, the local well-posedness of the solutions in the Besov spaces $C([0,T]; B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R}))$ with $s > \max\{\frac{1}{2}, \frac{1}{p}\}$ and $p, r \in [1, \infty]$ was established by using the transport equations theory and the Fourier analysis methods. Moreover, some precise blow-up criteria for the strong solutions to the system (1.1) are also derived.

Apparently, the integrable system (1.1) is a nature extension of the following cubic Fokas–Olver–Rosenau–Qiao (FORQ) equation (by taking $v = u$):

$$m_t + (u^2 - u_x^2)m_x + 2u_x m^2 + \gamma u_x = 0, \quad m = u - u_{xx}. \tag{1.4}$$

The FORQ equation was introduced by Fuchssteiner [23] and Olver and Rosenau [38] as a new generalization of integrable system by implementation a simple explicit algorithm based on the bi-Hamiltonian representation of the classical integrable system. The FORQ equation admits the Lax pair and the Cauchy problem may be solved by the inverse scattering transform method. In [25], Gui et al. proved that the FORQ equation with $\gamma = 0$ has single peakons given by $u(x,t) = \sqrt{\frac{3c}{2}}e^{-|x-ct|} (c > 0)$. Moreover, it has also the two-peakon solutions which can be formulated by

$$u(x,t) = \sqrt{\frac{3c_1}{2}}e^{-\left|x - c_1 t - \frac{3\sqrt{c_1 c_2}}{c_1 - c_2}e^{(c_1-c_2)t}\right|} + \sqrt{\frac{3c_2}{2}}e^{-\left|x - c_2 t - \frac{3\sqrt{c_1 c_2}}{c_1 - c_2}e^{(c_1-c_2)t}\right|},$$

where c_2, c_1 are positive constants satisfying $c_2 > c_1$. Later, Fu et al. [22] showed that the FORQ equation does not have any nontrivial smooth traveling wave solutions. Meanwhile, the authors investigated the local well-posedness of the FORQ equation in the Besov spaces $B^s_{p,r}(\mathbb{R})$ with $s > \max\{\frac{5}{2}, 2 + \frac{1}{p}\}$ and $p, r \in [1, \infty]$ by using the Littlewood–Paley theory and the transport equation theory which was first applied by Danchin [8]. Then, the well-posedness in the Besov space $B^s_{2,1}(\mathbb{R})$ with the critical index $s = \frac{5}{2}$ is also obtained. In [27], Himonas and Mantzavinos studied the well-posedness of (1.4) in Sobolev spaces $H^s(\mathbb{R}) (s > \frac{5}{2})$ in the sense of Hadamard, and they proved that the solution map is continuous but not uniformly continuous. In addition, the data-to-solution map is not uniformly continuous in the Sobolev spaces $H^s(\mathbb{R})$ if the regularity index s is less than $\frac{3}{2}$.

The other cubic integrable equation which is closely related to the FORQ equation is the celebrated Novikov equation

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}. \tag{1.5}$$

It was discovered by V. Novikov in a symmetry classification of nonlocal partial differential equations with quadratic or cubic nonlinearity [37]. After the Novikov equation was derived, many papers were devoted