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A sub-supersolution method for nonlinear elliptic singular systems with natural growth and some applications

José Carmona[a], Pedro J. Martínez-Aparicio[b], Antonio Suárez[c,*]

[a] *Dpto. de Matemáticas, Univ. de Almería, Ctra. Sacramento s/n, La Cañada de San Urbano, 04120 - Almería, Spain*

[b] *Dpto. de Matemática Aplicada y Estadística, Univ. Politécnica de Cartagena, 30202 - Murcia, Spain*

[c] *Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, 41012 - Sevilla, Spain*

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ABSTRACT

In this paper we give a sub-supersolution method for nonlinear elliptic singular systems with quadratic gradient whose model system is the following

$$\begin{cases} -\Delta u + v^{\beta}\dfrac{|\nabla u|^2}{u^{\alpha}} = f_1(x,u,v) & \text{in } \Omega, \\ -\Delta v + u^{\mu}\dfrac{|\nabla v|^2}{v^{\gamma}} = f_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 3)$, $\beta,\mu \geq 0$, $0 < \alpha,\gamma < 1$ and regular f_1, f_2 functions. Moreover, we apply it to prove existence of solution for some systems, including the classical Lotka–Volterra models with gradient terms. Specifically, we study the competition and the symbiotic Lotka–Volterra systems.

1. Introduction

The aim of this paper is to provide a sub-supersolution method for the following nonlinear elliptic singular system with natural growth

$$\begin{cases} -\Delta u + g_1(v)\dfrac{|\nabla u|^2}{u^{\alpha}} = f_1(x,u,v) & \text{in } \Omega, \\ -\Delta v + g_2(u)\dfrac{|\nabla v|^2}{v^{\gamma}} = f_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: jcarmona@ual.es (J. Carmona), pedroj.martinez@upct.es (P.J. Martínez-Aparicio), suarez@us.es (A. Suárez).

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where Ω is a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 3)$, $0 < \alpha, \gamma < 1$, the functions $g_1, g_2 \in C([0, +\infty))$ and $f_1, f_2 \in C(\overline{\Omega} \times [0, +\infty) \times [0, +\infty))$ verifying some general conditions detailed below.

Regarding the literature there are several papers about equations with quadratic gradient terms. The existence of solutions of the equation

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = a(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

for every function $a(x)$ in a given Lebesgue space has been systematically studied in [5,8,9] and references therein (in fact, for a more general nonlinear term $H(x, u, \nabla u)$ instead of $g(u)|\nabla u|^2$). They consider in the lower order term a continuous g in $\mathbb{R}$ which does not satisfy any growth restriction and the sign condition $g(s)s \geq 0$ for every $s \in \mathbb{R}$ is assumed. Thanks to the presence of the lower order term the Dirichlet problem associated to the equation is allowed to have finite energy weak solutions.

In [15,4] some of the above results were extended to the case of systems. Specifically, in [4] the authors study systems of elliptic equations with quadratic gradient. They consider a general system

$$\begin{cases} -\Delta u_i + H_i(x, u, \nabla u) = a_i(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \quad i = 1, \ldots, n \end{cases}$$

where $u = (u_1, \ldots, u_n)$, $a_i \in H^{-1}(\Omega)$ and the quadratic terms $H_i(x, u, \nabla u)$ satisfy a more general one-side condition than the sign condition, but in the case $H_i(x, u, \nabla u) = g_i(u)|\nabla u|^2$ this one-side hypothesis is equivalent to the sign condition. In their case g_i is continuous in $\mathbb{R}^n$ and they prove the existence of solution in the Sobolev space.

In the last years, Eq. (1.2) has attracted much attention by the presence of singular terms in front of the gradient, see [1,2,6] and references therein.

In [11] we proved that a sub-supersolution method works for equations of the form

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^\alpha} = f(\lambda, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and we apply it to different models.

In this paper we focus our attention to systems with quadratic gradient and singular terms as (1.1).

Let us mention that the sub-supersolution method is valid for semilinear systems, see for instance [13, 20]. In this case, when $g_1 \equiv g_2 \equiv 0$, the natural extension of the scalar definition of sub-supersolution depends on the monotonicity of the functions f_1 and f_2 with respect to v and u, respectively. A general definition was given in [13,20] where a pair of functions $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$, $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in H^1(\Omega) \cap L^\infty(\Omega)$ is called a sub-supersolution if

$$\begin{aligned} &\underline{u} \leq \overline{u}, \qquad \underline{v} \leq \overline{v} \quad \text{in } \Omega, \\ &\underline{u} \leq 0 \leq \overline{u}, \qquad \underline{v} \leq 0 \leq \overline{v} \quad \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} &-\Delta \underline{u} \leq f_1(x, \underline{u}, v), \qquad -\Delta \overline{u} \geq f_1(x, \overline{u}, v), \quad \forall v \in [\underline{v}, \overline{v}], \\ &-\Delta \underline{v} \leq f_2(x, u, \underline{v}), \qquad -\Delta \overline{v} \geq f_2(x, u, \overline{v}), \quad \forall u \in [\underline{u}, \overline{u}], \end{aligned}$$

where, given two ordered functions $z \leq w$, we have denoted

$$[z, w] := \{q \in L^\infty(\Omega) : z(x) \leq q(x) \leq w(x)\}$$

(see also [19] where it is proved the validity of the method for singular semilinear systems). Assuming the existence of a sub-supersolution, $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$, there exists a solution $(u, v) \in I \equiv [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ of the semilinear system (i.e. (1.1) with $g_1 \equiv g_2 \equiv 0$).